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Schouten identities for Feynman graph amplitudes; the Master Integrals for the two-loop massive sunrise graph

Ettore Remiddi[a, 1] and Lorenzo Tancredi[b, 2]

[a] *Dipartimento di Fisica, Università di Bologna and INFN, Sezione di Bologna, I-40126 Bologna, Italy*

[b] *Institut für Theoretische Physik, Universität Zürich, Wintherturerstrasse 190, CH-8057 Zürich, Switzerland*

Abstract

A new class of identities for Feynman graph amplitudes, dubbed Schouten identities, valid at fixed integer value of the dimension d is proposed. The identities are then used in the case of the two-loop sunrise graph with arbitrary masses for recovering the second-order differential equation for the scalar amplitude in $d = 2$ dimensions, as well as a chained set of equations for all the coefficients of the expansions in $(d-2)$. The shift from $d \approx 2$ to $d \approx 4$ dimensions is then discussed.

Key words: Feynman graphs, Multi-loop calculations, Schouten identities

[1] e-mail: ettore.remiddi@bo.infn.it
[2] e-mail: tancredi@physik.uzh.ch

1 Introduction

The Feynman integrals associated to the two-loop loop self-mass Feynman graph of Fig.(1), usually referred to as *sunrise*, have been widely studied in the literature within the framework of the integration by parts identities [1, 2], and it is by now well known that they can be expressed in terms of four Master Integrals (M.I.s), [3], which satisfy a system of four first-order coupled differential equations, [4] (equivalent to a single fourth-order differential equation for any of the Master Integrals). Several numerical approaches to the numerical solution of the equations with satisfactory degree of precision have been worked out, (see for instance [5]), but a complete treatment of the general case with three different masses in $d = 4$ dimensions is still missing.

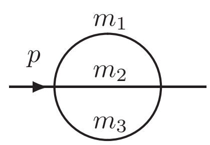

Figure 1: The two-loop sunrise.

In the equal mass case the number of independent Master Integrals reduces to two, so that the two by two first-order system of differential equations can be rewritten as a single second-order differential equation for one of the Master Integrals, say the full scalar amplitude (see below). In Ref. [6] it is shown how to build the analytic solution of that equation in terms of elliptic integrals, both for $d = 2$ and $d = 4$; the two cases, related by the Tarasov's shifting relations [7], are very similar, with the $d = 2$ case just marginally simpler than the $d = 4$ case. The analytic solution provides with the necessary information for writing out very precise and fast converging expansions for the accurate numerical evaluation of the two M.I.s [8].

More recently, an interesting paper [9] has shown, by using algebraic geometry arguments, that in $d = 2$ dimensions the full scalar amplitude satisfies a second-order differential equation also in the different mass case. The equation was then solved in [10] by suitably extending the method of [6]; let us observe here that the analytic solution of the second-order differential equation is equivalent to the analytic knowledge of two (of the four) Master Integrals of the sunrise with different masses.

The problem of extending the approach to $d = 4$, which is the physically relevant case, remains, as the straightforward use of the Tarasov's dimension-shifting relations is unfortunately not sufficient. Indeed, as will be shown in this paper, by explicitly working out the shifting relations one finds that any of the four Master Integrals at $d \approx 4$ dimensions can be expressed as a combination of *all* the four Master Integrals at $d \approx 2$ dimensions and of the first terms of their expansion in $(d - 2)$, while the results of [9] give only two of the four Master Integrals at exactly $d = 2$ dimensions, but no other information on the remaining Master Integrals and their expansion in $(d - 2)$.

In this paper we introduce a family of particular polynomials in the scalar products of the vectors occurring in the Feynman integrals, dubbed *Schouten polynomials*, which have the property of vanishing at some fixed integer value of the dimension d. By using those polynomials one can introduce an *ad hoc* set of amplitudes, from which one can at least in principle extract an independent set of new amplitudes which vanish in a non trivial way (see below) at that value of d (say at $d = N$ for definiteness). If those new amplitudes are expressed in terms of the previously chosen set of Master Integrals, their vanishing gives a set of relations between the Master Integrals, valid at $d = N$, which we call *Schouten identities.* Alternatively, one can introduce a new set of Master Integrals including as new Master Integrals some of the independent amplitudes vanishing at $d = N$, write the system of differential equations satisfied by the new set of Master Integrals and expand them recursively in powers of $(d - N)$ around $d = N$. As some of the new Master Integrals vanish at $d = N$, the system of equations takes a simpler block structure.

The pattern is very general, and applies in principle to the integrals of any Feynman graph. We work out explicitly the case of the sunrise amplitudes at $d = 2$ with different masses, finding the existence of two independent Schouten identities, *i.e.* of two independent relations between the usual Master Integrals at $d = 2$, or, which is the same, we can introduce a new set of Master Integrals, consisting of two "conventional" Master Integrals (say the full scalar amplitude and another M.I.) and two new Master Integrals vanishing at $d = 2$. The system of differential equations satisfied by the new set of Master Integrals can then be expanded in powers of $(d-2)$. At zeroth-order we find a two by two system for the two "conventional" M.I.s (the other two Master Integrals vanish), equivalent to the second-order equation found in [9], while at first-order in $(d-2)$ we find in particular two relatively simple equations for the first terms of the expansion of the two new M.I.s, in which the zeroth-orders of the two "conventional" M.I.s appear as non homogeneous known terms.

One can move from $d \approx 2$ to the physically more interesting $d \approx 4$ case by means of the Tarasov's shifting relations; it is found that for obtaining the zeroth-order term in $(d-4)$ of all the four M.I.s (of the old or of the new set) at $d \approx 4$ one needs, besides the zeroth-order term in $(d-2)$ of the two "old" M.I.s at $d \approx 2$, also the first term in $(d-2)$ of the new M.I.s.

The plan of the paper is as follows: in sec. 2 we introduce the Schouten polynomials for an arbitrary number of dimensions, while their applications to Feynman Amplitudes is discussed in sec. 3. In sec. 4 we show how, by using the Schouten Identities, a new set of Master Integrals can be found, whose differential equations in $d = 2$ take an easier block form and can be therefore re-casted (see sec. 5) as a second-order differential equation for one of the Masters. In sec. 6 we show how the results at $d \approx 4$ can be recovered from those at $d \approx 2$ through Tarasov's shifting relations. Finally, in sec. 7, which is somewhat pedagogical, we present a thorough treatment of the imaginary parts of the Master Integrals in $d = 2$ and $d = 4$ dimensions. Many lengthy formulas and some explicit derivations can be found in the Appendices at the end of the paper.

2 The Schouten Polynomials

As an introduction, let us recall that in $d = 4$ dimensions one cannot have more than 4 linearly independent vectors; indeed, given five vectors $v_\alpha, a_\mu, b_\nu, c_\rho, d_\sigma$ in four dimensions they are found to satisfy the following relation

$$v_\mu \epsilon(a,b,c,d) - a_\mu \epsilon(v,b,c,d) - b_\mu \epsilon(a,v,c,d) - c_\mu \epsilon(a,b,v,d) - d_\mu \epsilon(a,b,c,v) = 0 \ , \tag{2.1}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the usual Levi-Civita tensor with four indices, with $\epsilon_{1234} = 1, etc.$, and following the convention introduced in the program SCHOONSCHIP [11] we use

$$\epsilon(a,b,c,d) = \epsilon_{\mu\nu\rho\sigma} a_\mu b_\nu c_\rho d_\sigma \ . \tag{2.2}$$

Eq.(2.1) is known as the Schouten identity [12]; by squaring it, one gets a huge polynomial, of fifth-order in the scalar products of all the vectors. Due to Eq.(2.1), that polynomial vanishes in $d = 4$ dimensions (and, *a fortiori* for any integer dimension $d \leq 4$); note however that the polynomial does not vanish identically for any arbitrary value of the dimension; as Eq.(2.1) is valid only when $d \leq 4$, for $d > 4$ the polynomial is not bound to take a vanishing value.

As an extension (or rather a simplification) of Eq.(2.1), consider now the quantity

$$\epsilon(a,b) = \epsilon_{\mu\nu} a_\mu b_\nu \ , \tag{2.3}$$

where $\epsilon_{\mu\nu}$ is the Levi-Civita tensor with two indices (defined of course by $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$), and a_μ, b_ν are a couple of two-dimensional vectors. By squaring it, Eq.(2.3) gives at once

$$\epsilon^2(a,b) = a^2 b^2 - (a \cdot b)^2 \ , \tag{2.4}$$

where a^2, b^2 are the squared moduli of the vectors a_μ, b_ν and $(a \cdot b)$ their scalar product.
So far, all the quantities introduced in Eq.s(2.3,2.4) are in $d = 2$ dimensions. If the dimension d takes the value of any (non-vanishing) integer less than 2 (*i.e.* if $d = 1$), the *r.h.s.* of Eq.(2.3) vanishes, and so does the *r.h.s.* of Eq.(2.4) as well. At this point we define the Schouten Polynomial $P_2(d;a,b)$ as

$$P_2(d;a,b) = a^2 b^2 - (a \cdot b)^2 \ , \tag{2.5}$$

where the *r.h.s.* is formally the same *r.h.s.* of Eq.(2.4), but the two vectors a_μ, b_ν are assumed to be d-dimensional vectors, with continuous d. To emphasize that point, we have written d within the arguments of the Schouten Polynomial, even if d does not appear explicitly in the *r.h.s.* of Eq.(2.5). By the very definition, at integer non vanishing dimension $d < 2$ (*i.e.* at $d = 1$), $P_2(d; a, b)$ vanishes,

$$P_2(1; a, b) = 0 \ , \tag{2.6}$$

as can be also verified by an absolutely trivial explicit calculation.

Following the elementary procedure leading to Eq.(2.5), given in $d = 3$ dimensions any triplet of vectors a_μ, b_ν, c_ρ we consider

$$\epsilon(a, b, c) = \epsilon_{\mu\nu\rho} a_\mu b_\nu c_\rho \ , \tag{2.7}$$

where $\epsilon_{\mu\nu\rho}$ is the Levi-Civita tensor with three indices (defined as usual by $\epsilon_{123} = 1$ *etc.*) and then we evaluate its square

$$\epsilon^2(a, b, c) = a^2 b^2 c^2 - a^2(b \cdot c)^2 - b^2(a \cdot c)^2 - c^2(a \cdot b)^2 + 2(a \cdot b)(b \cdot c)(a \cdot c) \ . \tag{2.8}$$

We then define the Schouten Polynomial $P_3(d; a, b, c)$ as

$$P_3(d; a, b, c) = a^2 b^2 c^2 - a^2(b \cdot c)^2 - b^2(a \cdot c)^2 - c^2(a \cdot b)^2 + 2(a \cdot b)(b \cdot c)(a \cdot c) \ . \tag{2.9}$$

where again the *r.h.s.* is formally the same as in Eq.(2.8), but the three vectors a_μ, b_ν, c_ρ are assumed to be d-dimensional vectors, with continuous d. By construction, $P_3(d; a, b, c)$ vanishes at $d = 1$ and at $d = 2$,

$$\begin{aligned} P_3(1; a, b, c) &= 0 \ , \\ P_3(2; a, b, c) &= 0 \ . \end{aligned} \tag{2.10}$$

Needless to say, the procedure can be immediately iterated to any higher dimension, generating Schouten polynomials involving four vectors and vanishing in $d = 1, 2, 3$ dimensions, or involving five vectors and vanishing in $d = 1, 2, 3, 4$ dimensions, corresponding, up to a constant numerical factor, to the square of Eq.(2.1), *etc.*.

As it is apparent from the previous discussion, the Schouten polynomials generated by a given set of vectors are nothing but the Gram determinants of the corresponding vectors; we prefer to refer to them as Schouten polynomials to emphasize that they vanish in any integer dimension d less than the number of the vectors.

In the actual physical applications, as one is interested mainly in the $d \to 4$ limit of Feynman graph amplitudes, one can reach $d = 4$ starting from a different value of d and then moving to $d = 4$ by means of the Tarasov's shifting relations [7]. As the shift relates values of d differing by two units, the $d = 1$ Schouten identities, easily established for any amplitude in which at least two vectors occur, are of no use. The next simplest identities are at $d = 2$ and occur with any amplitude involving at least three vectors. That is the case of the two-loop self-mass graph (the sunrise), which we will study in this paper in the arbitrary mass case.

3 The Schouten Identities for the Sunrise graph

We discuss in this Section the use of the Schouten polynomial $P_3(d; a, b, c)$ in the case of the sunrise, the two-loop self-mass graph of Fig.(1).

The external momentum is p and the internal masses are m_1, m_2, m_3. We use the Euclidean metric, so that p^2 is positive when spacelike; sometimes we will use also $s = W^2 = -p^2$, so that the sunrise amplitudes develop an imaginary part when $\sqrt{s} = W > (m_1 + m_2 + m_3)$, the threshold of the Feynman graph. We write the propagators as

$$\begin{aligned} D_1 &= q_1^2 + m_1^2 \ , \\ D_2 &= q_2^2 + m_2^2 \ , \\ D_3 &= (p - q_1 - q_2)^2 + m_3^2 \ , \end{aligned} \tag{3.1}$$

and define the loop integration measure, in agreement with previous works, as:

$$\int \mathfrak{D}^d q = \frac{1}{C(d)} \int \frac{d^d q}{(2\pi)^{d-2}} \,, \tag{3.2}$$

with

$$C(d) = (4\pi)^{(4-d)/2} \Gamma\left(3 - \frac{d}{2}\right) \,, \tag{3.3}$$

so that

$$C(2) = 4\pi \quad \text{and} \quad C(4) = 1 \,. \tag{3.4}$$

With that definition the Tadpole $T(d, m)$ reads

$$T(d; m) = \int \mathfrak{D}^d q \, \frac{1}{q^2 + m^2} = \frac{m^{d-2}}{(d-2)(d-4)} \,. \tag{3.5}$$

In this paper we will use the "double" tadpoles

$$T(d; m_1, m_2) = \int \mathfrak{D}^d q_1 \, \mathfrak{D}^d q_2 \frac{1}{D_1 D_2} \,, \tag{3.6}$$

together with the similarly defined $T(d; m_1, m_3), T(d; m_2, m_3)$, and the four amplitudes

$$\begin{aligned}
S(d; p^2) &= \int \mathfrak{D}^d q_1 \, \mathfrak{D}^d q_2 \frac{1}{D_1 D_2 D_3} \,, \\
S_1(d; p^2) &= -\frac{d}{dm_1^2} S(d; p^2) = \int \mathfrak{D}^d q_1 \, \mathfrak{D}^d q_2 \frac{1}{D_1^2 D_2 D_3} \,, \\
S_2(d; p^2) &= -\frac{d}{dm_2^2} S(d; p^2) = \int \mathfrak{D}^d q_1 \, \mathfrak{D}^d q_2 \frac{1}{D_1 D_2^2 D_3} \,, \\
S_3(d; p^2) &= -\frac{d}{dm_3^2} S(d; p^2) = \int \mathfrak{D}^d q_1 \, \mathfrak{D}^d q_2 \frac{1}{D_1 D_2 D_3^2} \,.
\end{aligned} \tag{3.7}$$

All those amplitudes depend on the three masses m_1, m_2, m_3, even if the masses are not written explicitly in the arguments for simplicity. The four amplitudes are equal, when multiplied by an overall constant factor $(2\pi)^4$, to the four M.I.s used in [4]. $S(d; p^2)$, in particular, is the full scalar amplitude already referred to previously. Those amplitudes were chosen in [4] as M.I.s for the sunrise problem, and in the following they will be sometimes referred to as the "conventional" M.I.s .

We can now introduce the *Schouten amplitudes* defined, for arbitrary d, as

$$Z(d; n_1, n_2, n_3, p^2) = \int \mathfrak{D}^d q_1 \, \mathfrak{D}^d q_2 \frac{P_3(d; p, q_1, q_2)}{D_1^{n_1} D_2^{n_2} D_3^{n_3}} \,, \tag{3.8}$$

where the n_i are positive integer numbers and $P_3(d; p, q_1, q_2)$ is the Schouten polynomial defined in Eq.(2.9). The convergence of the integrals, for a given value of d, depends of course on the powers n_i, as the Schouten polynomial in the numerator contributes always with four powers of the loop momenta q_1 and q_2.

We are interested here in the $d = 2$ case. If the Schouten amplitude is convergent at $d = 2$, due to the second of Eq.s(2.10), it is also vanishing at $d = 2$, *i.e.* $Z(2; n_1, n_2, n_3, p^2) = 0$. Note that in the massive case all the integrals we are considering are *i.r.* finite, therefore the divergences can only be of *u.v.* nature.

As one can express any sunrise Feynman amplitude in terms of a valid set of M.I.s, we will write in the following a few Schouten amplitudes in terms of the "conventional" M.I.s given in Eq.s(3.7). A few explicit results are now listed:

$$\begin{aligned} Z_1(d;p^2) &= Z(d;1,2,2) \\ &= \frac{(d-1)}{12}\left[-(d-2)p^2+(d-3)(-2m_1^2+m_2^2+m_3^2))\right] S(d;p^2) \\ &- \frac{(d-1)}{6}(p^2+m_1^2)\ m_1^2 S_1(d;p^2) \\ &+ \frac{(d-1)}{12}(p^2-3m_1^2+m_2^2+3m_3^2)\ m_2^2 S_2(d;p^2) \\ &+ \frac{(d-1)}{12}(p^2-3m_1^2+3m_2^2+m_3^2)\ m_3^2 S_3(d;p^2) \\ &+ \frac{(d-1)(d-2)}{24}\left[T(d;m_1,m_2)+T(d;m_1,m_3)-2\,T(d;m_2,m_3)\right] \ , \end{aligned} \tag{3.9}$$

$$\begin{aligned} Z_2(d;p^2) &= Z(d;2,1,2,p^2) \\ &= \frac{(d-1)}{12}\left[-(d-2)p^2+(d-3)(m_1^2-2m_2^2+m_3^2)\right] S(d;p^2) \\ &+ \frac{(d-1)}{12}(p^2+m_1^2-3m_2^2+3m_3^2)\ m_1^2 S_1(d,p^2) \\ &- \frac{(d-1)}{6}(p^2+m_2^2)\ m_2^2 S_2(d;p^2) \\ &+ \frac{(d-1)}{12}(p^2+3m_1^2-3m_2^2+m_3^2)\ m_3^2 S_3(d;p^2) \\ &+ \frac{(d-1)(d-2)}{24}\left[T(d;m_1,m_2)-2\,T(d;m_1,m_3)+T(d;m_2,m_3)\right] \ , \end{aligned} \tag{3.10}$$

$$\begin{aligned} Z_3(d;p^2) &= Z(d;2,2,1,p^2) \\ &= \frac{(d-1)}{12}\left[-(d-2)p^2+(d-3)(m_1^2+m_2^2-2m_3^2)\right] S(d;p^2) \ , \\ &+ \frac{(d-1)}{12}(p^2+m_1^2+3m_2^2-3m_3^2)\ m_1^2 S_1(d;p^2) \\ &+ \frac{(d-1)}{12}(p^2+3m_1^2+m_2^2-3m_3^2)\ m_2^2 S_2(d;p^2) \\ &- \frac{(d-1)}{6}(p^2+m_3^2)\ m_3^2 S_3(d;p^2) \\ &+ \frac{(d-1)(d-2)}{24}\left[-2\,T(d;m_1,m_2)+T(d;m_1,m_3)+T(d;m_2,m_3)\right] \ , \end{aligned} \tag{3.11}$$

$$\begin{aligned} Z(d;2,2,2,p^2) &= -\frac{(d-1)(d-2)}{4} \\ &\times \left[(d-3)S(d;p^2)+m_1^2 S_1(d;p^2)+m_2^2 S_2(d;p^2)+m_3^2 S_3(d;p^2)\right] \ . \end{aligned} \tag{3.12}$$

Some comments are in order. Elementary power counting arguments give $N = 2(n_1+n_2+n_3)$ powers of the integration momenta in the denominator (independently of d) and, in $d = 2$ dimensions, all together eight powers in the numerator (see Eq.(3.8) for the definition of the integrals), so that the minimum value of N necessary to guarantee the convergence is $N = 10$. In the case of $Z(d;2,2,2,p^2)$ of Eq.(3.12) $N = 12$, more than the minimum required value $N = 10$; therefore the integrals in the loop momenta q_1, q_2 do converge, so that the vanishing of $P_3(d;p,q_1,q_2)$ in the numerator at $d = 2$ (and, as a matter of fact at $d = 1$ as well) does imply the vanishing of the whole amplitude. The explicit result, Eq.(3.12), shows indeed that the amplitude vanishes at $d = 1$ and $d = 2$, but that is due to an overall factor $(d-1)(d-2)$, so that

Eq.(3.12) does not give any useful information. This pattern − the vanishing at $d = 2$ of the amplitudes with $P_3(d; p, q_1, q_2)$ in the numerator and $N > 10$ due to the appearance of an overall factor $(d - 2)$ − is of general nature, and showed up in all the cases which we were able to check (needless to say, the algebraic complications increase quickly with the powers of the denominators).

The $Z_i(d; p^2)$, Eq.s(3.9,3.10,3.11), are more interesting; in this case, $N = 10$, which is the minimum value needed to guarantee convergence in $d = 2$ dimensions, so that those amplitudes are expected to vanish at $d = 2$ (and therefore also at $d = 1$) as a consequence of the vanishing of $P_3(d; p, q_1, q_2)$ at $d = 1, d = 2$, see Eq.s(2.10). The vanishing at $d = 1$ is trivially given by the overall factor $(d-1)$ (in $d = 1$ the minimum value of N to guarantee convergence is $N = 8$, while in the integrals we are now considering $N = 10$), but the vanishing at $d = 2$ is totally non trivial, providing new (and so far not known) relations between the four conventional M.I.s $S(d; p^2), S_i(d; p^2)$ at $d = 2$.

Any of the three amplitudes $Z_i(d; p^2)$ can obviously be obtained from the others by a suitable permutation of the three masses, as immediately seen from their explicit expression. When summing the three relations, one obtains

$$Z_1(d; p^2) + Z_2(d; p^2) + Z_3(d; p^2) = -\frac{(d-1)(d-2)}{4} p^2\ S(d; p^2)\ , \tag{3.13}$$

showing that at $d = 2$ they are not independent from each other; indeed, if one takes as input $Z_2(2; p^2) = 0$ and $Z_3(2; p^2) = 0$, the previous equation gives $Z_1(2; p^2) = 0$, showing that the condition $Z_1(2; p^2) = 0$ depends on the other two. When written explicitly, the vanishing of $Z_2(2; p^2) = 0$ and $Z_3(2; p^2) = 0$ reads

$$\begin{aligned} Z_2(2; p^2) = &-\frac{1}{12}(m_1^2 - 2m_2^2 + m_3^2) S(2; p^2) \\ &+ \frac{1}{12}(p^2 + m_1^2 - 3m_2^2 + 3m_3^2)\ m_1^2 S_1(2, p^2) \\ &- \frac{1}{6}(p^2 + m_2^2)\ m_2^2 S_2(2; p^2) \\ &+ \frac{1}{12}(p^2 + 3m_1^2 - 3m_2^2 + m_3^2)\ m_3^2 S_3(2; p^2) \\ &+ \frac{1}{96} \ln \frac{m_2^2}{m_1 m_3} \\ &= 0\ , \end{aligned} \tag{3.14}$$

$$\begin{aligned} Z_3(2; p^2) = &-\frac{1}{12}(m_1^2 + m_2^2 - 2m_3^2) S(2; p^2)\ , \\ &+ \frac{1}{12}(p^2 + m_1^2 + 3m_2^2 - 3m_3^2)\ m_1^2 S_1(2; p^2) \\ &+ \frac{1}{12}(p^2 + 3m_1^2 + m_2^2 - 3m_3^2)\ m_2^2 S_2(2; p^2) \\ &- \frac{1}{6}(p^2 + m_3^2)\ m_3^2 S_3(2; p^2) \\ &+ \frac{1}{96} \ln \frac{m_3^2}{m_1 m_2} \\ &= 0\ . \end{aligned} \tag{3.15}$$

The validity of identities Eq.s(3.14,3.15) in $d = 2$ has been checked with SecDec [13]. By using the above relations, which hold identically in p^2, m_1^2, m_2^2, m_3^2, one can express two of the conventional M.I.s in terms of the other two, showing that, at $d = 2$, there are in fact only two independent M.I.s. As can be seen from Eq.s(3.14,3.15), the relations between the M.I.s are not trivial (in particular, none of the M.I.s vanishes at $d = 2$; according to the definition Eq.(3.7) for space-like p they are all positive definite).

4 A New Set of Master Integrals

We have seen in the previous Section that the "conventional" M.I.s in $d = 2$ dimensions satisfy two independent conditions, written explicitly in Eq.s(3.14,3.15), so that two of them can be expressed as a combination of the other two, which can be taken as independent. On the other hand, it is known that in the equal mass limit the Sunrise has two independent M.I.s (in any dimension, including $d = 2$) so that no other independent conditions can exist. It can therefore be convenient to introduce a new set of M.I.s, formed by two "conventional" M.I.s , say $S(d;p^2), S_1(d;p^2)$ of Eq.(3.7), and two Schouten amplitudes, say $Z_2(d;p^2), Z_3(d;p^2)$ of Eq.s(3.10,3.11). The advantage of the choice is that two conditions at $d = 2$ take the simple form $Z_2(2;p^2) = 0,\ Z_3(2;p^2) = 0$. The actual choice of the new M.I.s satisfying the above criteria is of course not unique (a fully equivalent set could be for instance $S(d;p^2), S_2(d;p^2), Z_1(d;p^2), Z_2(d;p^2)$ *etc.*).

In the new basis of M.I.s, the two discarded conventional M.I.s are expressed as

$$\begin{aligned}
&P(p^2,m_1,m_2,m_3)\, m_2^2\, S_2(d;p^2) = \\
&\qquad \Big\{(m_1^2-m_2^2)\left[(d-3)(m_1^2+m_2^2-m_3^2)-p^2\right]-(d-2)p^2(p^2+m_3^2)\Big\} S(d;p^2) \\
&\qquad + P(p^2,m_2,m_1,m_3)\, m_1^2\, S_1(d;p^2) \\
&\qquad - \frac{8}{(d-1)}\left(p^2+m_3^2\right) Z_2(d;p^2) \\
&\qquad - \frac{4}{(d-1)}\left(p^2+3m_1^2-3m_2^2+m_3^2\right) Z_3(d;p^2) \\
&\qquad - (d-2)\left(m_1^2-m_2^2\right) T(d;m_1,m_2) \\
&\qquad - \frac{(d-2)}{2}\left(p^2-m_1^2+m_2^2+m_3^2\right) T(d;m_1,m_3) \\
&\qquad + \frac{(d-2)}{2}\left(p^2+m_1^2-m_2^2+m_3^2\right) T(d;m_2,m_3)\ ,
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
&P(p^2,m_1,m_2,m_3)\, m_3^2\, S_3(d;p^2) = \\
&\qquad \Big\{(m_1^2-m_3^2)\left[(d-3)(m_1^2-m_2^2+m_3^2)-p^2\right]-(d-2)p^2(p^2+m_2^2)\Big\} S(d;p^2) \\
&\qquad + P(p^2,m_3,m_1,m_2)\, m_1^2\, S_1(d;p^2) \\
&\qquad - \frac{4}{(d-1)}\left(p^2+3m_1^2+m_2^2-3m_3^2\right) Z_2(d;p^2) \\
&\qquad - \frac{8}{(d-1)}\left(p^2+m_2^2\right) Z_3(d;p^2) \\
&\qquad - \frac{(d-2)}{2}\left(p^2-m_1^2+m_2^2+m_3^2\right) T(d;m_1,m_2) \\
&\qquad - (d-2)\left(m_1^2-m_3^2\right) T(d;m_1,m_3) \\
&\qquad + \frac{(d-2)}{2}\left(p^2+m_1^2+m_2^2-m_3^2\right) T(d;m_2,m_3)\ ,
\end{aligned} \tag{4.2}$$

where $P(p^2,m_1,m_2,m_3)$ is the polynomial

$$\begin{aligned}
P(p^2,m_1,m_2,m_3) &= p^4 + 2(m_2^2+m_3^2-m_1^2)p^2 \\
&\quad - 3m_1^4 + m_2^4 + m_3^4 + 2m_1^2m_2^2 + 2m_1^2m_3^2 - 2m_2^2m_3^2\ .
\end{aligned} \tag{4.3}$$

Note that $P(p^2,m_1,m_2,m_3)$, which is symmetric in the last two arguments,

$$P(p^2,m_1,m_2,m_3) = P(p^2,m_1,m_3,m_2)\ , \tag{4.4}$$

occurs with different arguments in different places.

By substituting the above expressions in the differential equations for the conventional M.I.s as given, for instance, in Ref. [4], one obtains the new equations

$$
\begin{aligned}
P(p^2,m_1,m_2,m_3)\, p^2 \frac{d}{dp^2}\, S(d;p^2) = {} & (p^2+m_1^2)\Big[(p^2-m_1^2+m_2^2+m_3^2) \\
& + (d-2)(p^2+m_1^2-m_2^2-m_3^2)\Big]\, S(d;p^2) \\
& - Q(p^2,m_1,m_2,m_3)\, m_1^2\, S_1(d;p^2) \\
& + \frac{4}{(d-1)}(3p^2+3m_1^2+m_2^2-m_3^2)\, Z_2(d;p^2) \\
& + \frac{4}{(d-1)}(3p^2+3m_1^2-m_2^2+m_3^2)\, Z_3(d;p^2) \\
& + \frac{(d-2)}{2}(p^2+m_1^2-m_2^2+m_3^2)\, T(d;m_1,m_2) \\
& + \frac{(d-2)}{2}(p^2+m_1^2+m_2^2-m_3^2)\, T(d;m_1,m_3) \\
& - (d-2)\,(p^2+m_1^2)\, T(d;m_2,m_3)\ , \qquad (4.5)
\end{aligned}
$$

$$
\begin{aligned}
& D(p^2,m_1,m_2,m_3)P(p^2,m_1,m_2,m_3)\, p^2 \frac{d}{dp^2}\, S_1(d;p^2) = \\
& \quad \Big[\frac{(d-2)^2}{2}\, \big(p^2+m_1^2-m_2^2-m_3^2\big)\, P_{10}^{(2)}(p^2,m_1,m_2,m_3) \\
& \quad - (d-2)\, P_{10}^{(1)}(p^2,m_1,m_2,m_3) - P_{10}^{(0)}(p^2,m_3,m_1,m_2)\Big] S(d;p^2) \\
& \quad + \left[\frac{(d-2)}{2}\, P_{11}^{(1)}(p^2,m_1,m_2,m_3) - P_{11}^{(0)}(p^2,m_1,m_2,m_3)\right]\, S_1(d;p^2) \\
& \quad + \frac{4(d-3)}{(d-1)}\Big[P_{12}^{(0)}(p^2,m_1,m_2,m_3)\, Z_2(d;p^2) + P_{12}^{(0)}(p^2,m_1,m_3,m_2)\, Z_3(d;p^2)\Big] \\
& \quad + \frac{(d-2)}{4}\, \left[\frac{(d-2)}{m_1^2}\, P_{14}^{(2)}(p^2,m_1,m_2,m_3) - 2\, P_{14}^{(1)}(p^2,m_1,m_2,m_3)\right]\, T(d;m_1,m_2) \\
& \quad + \frac{(d-2)}{4}\, \left[\frac{(d-2)}{m_1^2}\, P_{14}^{(2)}(p^2,m_1,m_3,m_2) - 2\, P_{14}^{(1)}(p^2,m_1,m_3,m_2)\right]\, T(d;m_1,m_3) \\
& \quad - \frac{(d-2)}{2}\, \Big[(d-2)\, P_{10}^{(2)}(p^2,m_1,m_2,m_3) \\
& \qquad\quad - \Big(P_{14}^{(1)}(p^2,m_1,m_2,m_3) + P_{14}^{(1)}(p^2,m_1,m_3,m_2)\Big)\, \Big]\, T(d;m_2,m_3)\ , \qquad (4.6)
\end{aligned}
$$

$$
\begin{aligned}
P(p^2,m_1,m_2,m_3)\, p^2 \frac{d}{dp^2}\, Z_2(d;p^2) = {} & p^2\, \frac{(d-1)(d-2)}{8} \Big[2\left(m_1^2-m_2^2\right)\left(p^2+m_1^2+m_2^2-m_3^2\right) \\
& + (d-2)\left(p^2+m_1^2-m_2^2-m_3^2\right)\left(p^2+m_1^2-m_2^2+m_3^2\right) \Big]\, S(d;p^2) \\
& - p^2\, \frac{(d-1)(d-2)}{4}\, P(p^2,m_2,m_1,m_3)\, m_1^2\, S_1(d;p^2) \\
& + \frac{(d-2)}{2}\, P_{22}(p^2,m_1,m_2,m_3)\, Z_2(d;p^2) \\
& + p^2\,(d-2)\, \left(p^2+3m_1^2-3m_2^2+m_3^2\right)\, Z_3(d;p^2) \\
& + p^2\, \frac{(d-1)(d-2)^2}{4}\, \left(m_1^2-m_2^2\right)\, T(d;m_1,m_2) \\
& + p^2\, \frac{(d-1)(d-2)^2}{8}\, \left(p^2-m_1^2+m_2^2+m_3^2\right)\, T(d;m_1,m_3) \\
& - p^2\, \frac{(d-1)(d-2)^2}{8}\, \left(p^2+m_1^2-m_2^2+m_3^2\right)\, T(d;m_2,m_3)\ ,
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
P(p^2,m_1,m_2,m_3)\, p^2 \frac{d}{dp^2}\, Z_3(d;p^2) = {} & p^2\, \frac{(d-1)(d-2)}{8} \Big[2\left(m_1^2-m_3^2\right)\left(p^2+m_1^2-m_2^2+m_3^2\right) \\
& + (d-2)\left(p^2+m_1^2-m_2^2-m_3^2\right)\left(p^2+m_1^2+m_2^2-m_3^2\right) \Big]\, S(d;p^2) \\
& - p^2\, \frac{(d-1)(d-2)}{4}\, P(p^2,m_3,m_1,m_2)\, m_1^2\, S_1(d;p^2) \\
& + p^2\,(d-2)\, \left(p^2+3m_1^2+m_2^2-3m_3^2\right)\, Z_2(d;p^2) \\
& + \frac{(d-2)}{2}\, P_{22}(p^2,m_1,m_3,m_2)\, Z_3(d;p^2) \\
& + p^2\, \frac{(d-1)(d-2)^2}{8}\, \left(p^2-m_1^2+m_2^2+m_3^2\right)\, T(d;m_1,m_2) \\
& + p^2\, \frac{(d-1)(d-2)^2}{4}\, \left(m_1^2-m_3^2\right)\, T(d;m_1,m_3) \\
& - p^2\, \frac{(d-1)(d-2)^2}{8}\, \left(p^2+m_1^2+m_2^2-m_3^2\right)\, T(d;m_2,m_3)\ .
\end{aligned} \tag{4.8}
$$

In the above equations,

$$
\begin{aligned}
D(p^2,m_1,m_2,m_3) = {} & (p^2+(m_1+m_2+m_3)^2)(p^2+(m_1-m_2+m_3)^2) \\
& (p^2+(m_1+m_2-m_3)^2)(p^2+(m_1-m_2-m_3)^2)
\end{aligned} \tag{4.9}
$$

is the product of all the threshold and pseudo-threshold factors already present in [4],

$$
\begin{aligned}
Q(p^2,m_1,m_2,m_3) = {} & -(m_1+m_2+m_3)(m_1-m_2+m_3)(m_1+m_2-m_3)(m_1-m_2-m_3) \\
& + 2\,p^2\left(m_1^2+m_2^2+m_3^2\right) + 3\,p^4\,,
\end{aligned} \tag{4.10}
$$

while $P(p^2,m_1,m_2,m_3)$ is the polynomial previously defined in Eq.(4.3). Finally the $P_{ij}^{(n)}(p^2,m_1,m_2,m_3)$ are also polynomials depending on p^2 and the masses; their explicit (and sometimes lengthy expression) is given in Appendix A. Note that a same polynomial can occur in different equations with a different permutation of the masses within its arguments.

We want to stress here an important aspect of the last two equations, Eq.(4.7,4.8), namely the presence of an overall factor $(d-2)$ in the *r.h.s.*, which plays an important role in the expansion in powers of $(d-2)$ discussed in the next Subsection.

4.1 The expansion of the Equations around $d = 2$

Let us start off by expanding all M.I.s in powers of $(d-2)$ around $d = 2$,

$$
\begin{aligned}
S(d;p^2) &= S(2;p^2) + (d-2)S^{(1)}(2,p^2) + ... \\
S_1(d;p^2) &= S_1(2;p^2) + (d-2)S_1^{(1)}(2,p^2) + ... \\
Z_2(d;p^2) &= Z_2(2;p^2) + (d-2)Z_2^{(1)}(2,p^2) + ... \\
Z_3(d;p^2) &= Z_3(2;p^2) + (d-2)Z_3^{(1)}(2,p^2) + ...
\end{aligned} \tag{4.11}
$$

Due to the overall factor $(d-2)$ in the *r.h.s.*, at 0th order in $(d-2)$ the differential equations Eq.s(4.7,4.8) become

$$
\begin{aligned}
\frac{d}{dp^2}\, Z_2(2;p^2) &= 0 \\
\frac{d}{dp^2}\, Z_3(2;p^2) &= 0 \ ,
\end{aligned} \tag{4.12}
$$

showing that $Z_2(2;p^2), Z_3(2;p^2)$ must be constants. But we know from Eq.s(3.14,3.15) the actual value of that constant (the two functions vanish identically, $Z_2(2;p^2) = 0,\ Z_3(2;p^2) = 0$), so that at 0th order in $(d-2)$ the differential equations Eq.(4.5), Eq.(4.6) for $S(2;p^2), S_1(2;p^2)$ become

$$
\begin{aligned}
P(p^2,m_1,m_2,m_3)\, p^2 \frac{d}{dp^2}\, S(2;p^2) &= \left(p^2+m_1^2\right)\, \left(p^2-m_1^2+m_2^2+m_3^2\right)\, S(2;p^2) \\
&- Q(p^2,m_1,m_2,m_3) m_1^2\, S_1(2;p^2) \\
&+ \frac{1}{8}\Big[\, \left(p^2+m_1^2\right) \ln\frac{m_1^2}{m_2 m_3} + \left(m_2^2-m_3^2\right) \ln\frac{m_3}{m_2} \Big] \ ,
\end{aligned} \tag{4.13}
$$

$$
\begin{aligned}
D(p^2,m_1,m_2,m_3)&P(p^2,m_1,m_2,m_3)\, p^2 \frac{d}{dp^2}\, S_1(2;p^2) = \\
&- P_{10}^{(0)}(p^2,m_1,m_2,m_3)\, S(2,p^2) - P_{11}^{(0)}(p^2,m_1,m_2,m_3)\, S_1(2,p^2) \\
&- \frac{1}{8}\Big[P_{14}^{(1)}(p^2,m_1,m_2,m_3) \ln\frac{m_1}{m_3} + P_{14}^{(1)}(p^2,m_1,m_3,m_2) \ln\frac{m_1}{m_2} \\
&- \frac{p^2}{m_1^2}\, P^2(p^2,m_1,m_2,m_3) \Big] \ ,
\end{aligned} \tag{4.14}
$$

completely decoupled, obviously, from the (trivial) equations for $Z_2(2;p^2), Z_3(2;p^2)$. (See the previous Section and Appendix A for the explicit expression of the polynomials.)

Going now one order higher in the expansion, one finds that the first-order terms in $(d-2)$ of the $Z_i(d;p^2)$ satisfy the equations

$$
\begin{aligned}
P(p^2,m_1,m_2,m_3)\, \frac{d}{dp^2}\, Z_2^{(1)}(2;p^2) &= \frac{1}{4}(m_1^2-m_2^2)(p^2+m_1^2+m_2^2-m_3^2)\, S(2;p^2) \\
&- \frac{1}{4} P(p^2,m_2,m_1,m_3)\, m_1^2\, S_1(2;p^2) \\
&+ \frac{1}{32}\Big[(p^2+m_3^2) \ln\frac{m_1}{m_2} + (m_1^2-m_2^2) \ln\frac{m_1 m_2}{m_3^2} \Big] \ ,
\end{aligned} \tag{4.15}
$$

$$
\begin{aligned}
P(p^2,m_1,m_2,m_3)\, \frac{d}{dp^2}\, Z_3^{(1)}(2;p^2) &= \frac{1}{4}(m_1^2-m_3^2)(p^2+m_1^2-m_2^2+m_3^2)\, S(2;p^2) \\
&- \frac{1}{4} P(p^2,m_3,m_1,m_2)\, m_1^2\, S_1(2;p^2) \\
&+ \frac{1}{32}\Big[(p^2+m_2^2) \ln\frac{m_1}{m_3} + (m_1^2-m_3^2) \ln\frac{m_1 m_3}{m_2^2} \Big] \ .
\end{aligned} \tag{4.16}
$$

It is to be noted that $Z_2^{(1)}(2;p^2), Z_3^{(1)}(2;p^2)$ do not appear in the *r.h.s.* of Eq.s(4.15,4.16), which contains only $S(2;p^2)$ and $S_1(2;p^2)$, to be considered known once Eq.s(4.13,4.14) for the $0th$ orders in $(d-2)$ have been solved. Eq.s(4.15,4.16), indeed, are absolutely trivial when considered as differential equations, as they contain only the derivatives of $Z_2^{(1)}(2;p^2), Z_3^{(1)}(2;p^2)$, and can therefore be solved by a simple quadrature.

Knowing $Z_2^{(1)}(2;p^2), Z_3^{(1)}(2;p^2)$, one can move to the differential equations for $S^{(1)}(2;p^2), S_1^{(1)}(2;p^2)$ (which we don't write here for the sake of brevity); they involve $Z_2^{(1)}(2;p^2), Z_3^{(1)}(2;p^2)$ as known inhomogeneous terms, and form again a closed set of two differential equations, decoupled from the equations for the other two M.I.s, as at $0th$ order in $(d-2)$.

Thanks to the overall factor $(d-2)$ in the *r.h.s.* of Eq.s(4.7,4.8), the pattern – a quadrature for $Z_2^{(k)}(2;p^2), Z_3^{(k)}(2;p^2)$ and a closed set of two differential equations for $S^{(k)}(2;p^2), S_1^{(k)}(2;p^2)$ – is completely general, and can be iterated, at least in principle, up to any required order k in $(d-2)$.

5 Second-order Differential Equation for $S(d;p^2)$

We go back now to the system of differential equations Eq.s(4.5,4.6), for obtaining a second-order differential equation for $S(d;p^2)$. We can use Eq.(4.5) in order to express $S_1(d;p^2)$ in function of $S(d;p^2)$ and of its derivative, $dS(d;p^2)/dp^2$. By substituting this expression into Eq.(4.6) we can then derive a second-order differential equation for $S(d;p^2)$ only, which however still contains $Z_2(d;p^2)$ and $Z_3(d;p^2)$ in the inhomogeneous part:

$$
\begin{aligned}
A_1(p^2,m_1,m_2,m_3) \left(\frac{d}{dp^2}\right)^2 S(d;p^2) &+ \left[A_2^{(0)}(p^2,m_1,m_2,m_3) + (d-2)\, A_2^{(1)}(p^2,m_1,m_2,m_3)\right] \frac{d}{dp^2} S(d;p^2) \\
&+ (d-3)\left[A_3^{(0)}(p^2,m_1,m_2,m_3) + (d-2)\, A_3^{(1)}(p^2,m_1,m_2,m_3)\right] S(d;p^2) \\
&+ \frac{(d-3)}{(d-1)} \left[A_4(p^2,m_1,m_2,m_3)\, Z_2(d;p^2) + A_4(p^2,m_1,m_3,m_2)\, Z_3(d;p^2)\right] \\
&+ (d-2) \left[A_5^{(1)}(p^2,m_1,m_2,m_3) + (d-2)\, A_5^{(2)}(p^2,m_1,m_2,m_3)\right] T(d;m_1,m_2) \\
&+ (d-2) \left[A_5^{(1)}(p^2,m_1,m_3,m_2) + (d-2)\, A_5^{(2)}(p^2,m_1,m_3,m_2)\right] T(d;m_1,m_3) \\
&+ (d-2) \left[A_5^{(1)}(p^2,m_2,m_3,m_1) + (d-2)\, A_5^{(2)}(p^2,m_2,m_3,m_1)\right] T(d;m_2,m_3) \\
&= 0\,,
\end{aligned}
\tag{5.1}
$$

where $A_1(p^2,m_1,m_2,m_3) = p^2\, D(p^2,m_1^2,m_2^2,m_3^2)\, P(p^2,m_1,m_2,m_3)$, with $D(p^2,m_1^2,m_2^2,m_3^2)$ and $P(p^2,m_1,m_2,m_3)$ being the usual polynomials defined by Eq.s(4.3,4.9). The $A_j^{(n)}(p^2,m_1,m_2,m_3)$ are also polynomials which depend on the three masses and on p^2, but *do not* depend on the dimensions d. Their explicit expressions, as usual quite lengthy, can be found in Appendix B.

The equation above is exact in d but contains, besides $S(d;p^2)$ and its derivatives, also $Z_2(d;p^2)$ and $Z_3(d;p^2)$ as inhomogeneous terms. Nevertheless, recalling once more that $Z_2(2;p^2) = Z_3(2;p^2) = 0$, we can expand Eq.(5.1) in powers of $(d-2)$ and obtain at leading order in $(d-2)$ a second-order differential equation for $S(2;p^2)$ *only*:

$$
\begin{aligned}
A_1(p^2,m_1,m_2,m_3) \left(\frac{d}{dp^2}\right)^2 S(2;p^2) &+ A_2^{(0)}(p^2,m_1,m_2,m_3) \left(\frac{d}{dp^2}\right) S(2;p^2) \\
&- A_3^{(0)}(p^2,m_1,m_2,m_3)\, S(2;p^2) + \frac{1}{4}\left[A_5^{(2)}(p^2,m_1,m_2,m_3)\right. \\
&+ A_5^{(2)}(p^2,m_1,m_3,m_2) + A_5^{(2)}(p^2,m_2,m_3,m_1) \\
&\left. + A_5^{(1)}(p^2,m_1,m_2,m_3)\, \ln\left(\frac{m_1}{m_3}\right) + A_5^{(1)}(p^2,m_1,m_3,m_2)\, \ln\left(\frac{m_1}{m_2}\right)\right] = 0\,,
\end{aligned}
\tag{5.2}
$$

where we made use of the relation Eq.(B.8) of Appendix B. We compared Eq.(5.2) with the second-order differential equation derived in [9], finding perfect agreement. Eq.(5.2) has been solved in reference [10] in terms of one-dimensional integrals over elliptic integrals.

Upon inserting the result in Eq.(4.13) one can obtain $S_1(2;p^2)$ in terms of $S(2;p^2)$ and $dS(2;p^2)/dp^2$. Inserting then $S(2;p^2)$ and $S_1(2;p^2)$ in Eq.s(4.15, 4.16), one obtains by quadrature the first-order terms, $Z_2^{(1)}(2;p^2)$ and $Z_3^{(1)}(2;p^2)$, of the expansion in $(d-2)$ of $Z_2(d;p^2)$ and $Z_3(d;p^2)$.

Having these results on hand, we can now consider the first-order in $(d-2)$ of the Eq.(5.1), which is now a second-order differential equation for $S^{(1)}(2;p^2)$ only, with known inhomogeneous terms (namely $S(2;p^2)$, $Z_2^{(1)}(2;p^2)$ and $Z_3^{(1)}(2;p^2)$). Proceeding in this way, at least in principle, the whole procedure can be iterated up to any required order in $(d-2)$.

6 Shifting relations from $d=2$ to $d=4$ dimensions

In the previous Sections we have shown how to use the Schouten identities for writing the differential equations for the M.I.s of the massive sunrise at $d=2$ dimensions in block form, and outlined the procedure for obtaining iteratively all the coefficients of the expansion in $(d-2)$ of the four M.I.s starting from a second-order differential equation for $S(2;p^2)$, the leading term of the expansion.

The physically interesting case corresponds however to the expansion of the M.I.s for $d \approx 4$; we have therefore to convert the information given by the expansion at $d \approx 2$ in useful information at $d \approx 4$.

As it is well known, quite in general one can relate any Feynman integral evaluated in d dimensions to the very same integral evaluated in $(d-2)$ dimensions by means of the Tarasov's shifting relation [7]. This dimensional shift is achieved by acting on the Feynman integral with a suitable combination of derivatives with respect to the internal masses. In the case of the "conventional " M.I.s of the sunrise graph, as defined in Eq.(3.7), the shifting relations read:

$$
\begin{aligned}
S(d-2;p^2) &= \frac{2^2}{(d-6)}\,\Delta\, S(d;p^2)\,, \\
S_i(d-2;p^2) &= \frac{2^2}{(d-6)}\,\Delta\, S_i(d;p^2)\,, \qquad i=1,2,3\,,
\end{aligned}
\tag{6.1}
$$

where the differential operator Δ takes the form:

$$
\Delta = \frac{\partial}{\partial m_1^2}\frac{\partial}{\partial m_2^2} + \frac{\partial}{\partial m_1^2}\frac{\partial}{\partial m_3^2} + \frac{\partial}{\partial m_2^2}\frac{\partial}{\partial m_3^2}\,. \tag{6.2}
$$

Carrying out the derivatives in the integral representation for the four M.I.s of Eq.(3.7), one obtains a combination of integrals which are still related to the sunrise graph. They can be expressed in terms of the full set of M.I.s in d dimensions (by full set we mean the four M.I.s and the tadpoles); one obtains in that way a set of four equations which explicitly relate the four M.I.s of the sunrise graph evaluated in $(d-2)$ dimensions to suitable combinations of the same integrals (and of the tadpoles) evaluated in d dimensions. In that *direct* form the shifting relations would be of no practical use in our case, as they might give the M.I.s at $(d-2) \approx 2$ in terms of those (less known) at $d \approx 4$.

It is however straightforward to invert the system and, in this way, to obtain the *inverse* shifting relations, expressing the four M.I.s in $d+2 \approx 4$ dimensions in function of those in $d \approx 2$ dimensions. In addition, we can also use Eq.s(4.1,4.2) for expressing $S_2(d;p^2)$ and $S_3(d;p^2)$, in terms of $S(d;p^2)$, $S_1(d;p^2)$ and $Z_2(d;p^2)$, $Z_3(d;p^2)$. As a result one arrives at expressing any of the four "conventional" M.I.s $S(d{+}2,p^2)$, $S_i(d{+}2;p^2)$, $i=1,2,3$, as a linear combination (whose coefficients depend – and in a non trivial way – on d and the kinematical variables of the problem) of the "new" M.I.s $S(d;p^2)$, $S_1(d;p^2)$, $Z_2(d;p^2)$ and $Z_3(d;p^2)$ (and the tadpoles). Indicating for simplicity the four "conventional" M.I.s with $M_i(d)$ and with $N_j(d)$ the four "new" M.I.s and the tadpoles, and ignoring for ease of notation all the kinematical variables, the *inverse* shifting relations can be written as

$$
M_i(d+2) = \sum_j C_{i,j}(d) N_j(d)\ . \tag{6.3}
$$

Given a relation of the form

$$F(d+2) = G(d) \ ,$$

by expanding around $d = 2$ one has, quite in general

$$F(d+2) = \sum_{n=r}^{p} (d-2)^n \ F^{(n)}(4) \ ,$$
$$G(d) = \sum_{n=r}^{p} (d-2)^n \ G^{(n)}(2) \ ,$$

where r, the first value of the summation index, can be negative (as it is the case in a Laurent expansion), so that

$$F^{(n)}(4) = G^{(n)}(2) \ .$$

In the case of the *inverse* shift Eq.(6.3), one has that the coefficients of the expansion of the "conventional" M.I.s in $(d-4)$ for $d \approx 4$ are completely determined by those of the expansion in $(d-2)$ for $d \approx 2$ of the "new" M.I.s, discussed in the previous Sections, and of the tadpoles (expanding around $d = 2$ the two sides of Eq.(6.3) requires also the expansion of the coefficients $C_{i,j}(d)$, but that is not a problem once the *inverse* shift has been written down explicitly).

The explicit formulas of the *direct* or *inverse* shifting relations are easily obtained but very lengthy and we decided not to include them entirely here for the sake of brevity. For what follows, it is sufficient to discuss only the general features of one of the *inverse* shifting relations, namely the relation expressing $S(d+2;p^2)$ in terms of $S(d;p^2)$, $S_1(d;p^2)$ and $Z_2(d;p^2)$, $Z_3(d;p^2)$. Keeping for simplicity only the leading term of the expansion in $(d-2)$ of the coefficients we find:

$$\begin{aligned} S(d+2;p^2) = & \left[\, C(p^2,m_1,m_2,m_3) + \mathcal{O}\left(d-2\right) \right] S(d;p^2) \\ & + \left[\, C_1(p^2,m_1,m_2,m_3) + \mathcal{O}\left(d-2\right) \right] S_1(d;p^2) \\ & + \left[\frac{1}{d-2}\ C_2(p^2,m_1,m_2,m_3) + \mathcal{O}\left(1\right) \right] Z_2(d;p^2) \\ & + \left[\frac{1}{d-2}\ C_3(p^2,m_1,m_2,m_3) + \mathcal{O}\left(1\right) \right] Z_3(d;p^2) \\ & + \left[C_4^{(0)}(p^2,m_1,m_2,m_3) + \mathcal{O}\left(d-2\right) \right] T(d;m_1,m_2) \\ & + \left[C_5^{(0)}(p^2,m_1,m_2,m_3) + \mathcal{O}\left(d-2\right) \right] T(d;m_1,m_2) \\ & + \left[C_6^{(0)}(p^2,m_1,m_2,m_3) + \mathcal{O}\left(d-2\right) \right] T(d;m_1,m_2) \, . \end{aligned} \tag{6.4}$$

In the formula above the $C(p^2,m_1,m_2,m_3)$, $C_i(p^2,m_1,m_2,m_3)$, are ratios of suitable polynomials which, as usual, depend on p^2 and on the three masses but, most importantly, *do not* depend on the dimensions d. The explicit expressions for $C(p^2,m_1,m_2,m_3)$, $C_1(p^2,m_1,m_2,m_3)$, $C_2(p^2,m_1,m_2,m_3)$ and $C_3(p^2,m_1,m_2,m_3)$, which will also be used in the following, can be found in Appendix C, Eq.s(C.2-C.5). Note anyway that:

$$C_3(p^2,m_1,m_2,m_3) = C_2(p^2,m_1,m_3,m_2) \, .$$

By writing the expansion of $S(d+2;p^2)$ at $d \approx 2$ as

$$S(d+2;p^2) = \sum_n S^{(n)}(4;p^2)(d-2)^n \, , \tag{6.5}$$

and then expanding Eq.(6.4) at $d \approx 2$, one recovers the expression of the coefficients $S^{(n)}(4;p^2)$ in terms of the coefficients of the expansion of the four M.I.s and the tadpoles in $(d-2)$.

A few observations are in order. Eq.(6.4) exhibits an explicit pole in $1/(d-2)$ only in the coefficients of $Z_2(d;p^2)$ and $Z_3(d;p^2)$; recalling once more that at $d = 2$ both $Z_2(2;p^2)$ and $Z_3(2;p^2)$ are identically zero, see Eq.s(3.14,3.15), it is clear that these poles will not generate any singularity of $S(d;p^2)$ as $d \to 2$. On the other hand, the tadpoles in the *r.h.s.* of Eq.(6.4) do generate polar singularities of $S(d+2;p^2)$; recalling Eq.s(3.5,3.6) and by using the lengthy explicit form of the coefficients (which we did not write for brevity) one finds immediately

$$\begin{aligned} S^{(-2)}(4;p^2) &= -\frac{(m_1^2+m_2^2+m_3^2)}{8} , \\ S^{(-1)}(4;p^2) &= \frac{1}{32}\Big[p^2 + 6\left(m_1^2+m_2^2+m_3^2\right)\Big] \\ &\quad - \frac{1}{8}\Big[m_1^2 \ln(m_1^2) + m_2^2 \ln(m_2^2) + m_3^2 \ln(m_3^2)\Big], \end{aligned} \tag{6.6}$$

formulas already known for a long time in the literature [4].

As a second observation, let us look at the zeroth-order term $S^{(0)}(4;p^2)$ of $S(d;p^2)$ in $(d-4)$, *i.e.* the zeroth-order term in $(d-2)$ of Eq.(6.4). We have already commented the apparent polar singularity $1/(d-2)$ in the coefficients of $Z_2(d;p^2)$ and $Z_3(d;p^2)$, actually absent because $Z_2(2;p^2)$ and $Z_3(2;p^2)$ are both vanishing. But due to the presence of the $1/(d-2)$ polar factor, in order to recover the zeroth-order term $S^{(0)}(4;p^2)$, one needs, besides $S(2;p^2), S_1(2;p^2)$, also the first-order of the corresponding expansion of $Z_2(d;p^2)$ and $Z_3(d;p^2)$, namely $Z_2^{(1)}(2;p^2)$ and $Z_3^{(1)}(2;p^2)$ – obtained, in our approach, from the systematic expansion of the differential equations, see Eq.s(4.15,4.16) or Section 5.

The complete expression of $S^{(0)}(4;p^2)$, which is rather cumbersome, is given by Eq.(C.1) of Appendix C. The corresponding formulas for the other three M.I.s, i.e. the $S_i(d+2;p^2)$, can be obtained directly from the authors.

7 The imaginary parts of the Master Integrals.

In this Section, which is somewhat pedagogical, we discuss the relationship between the imaginary parts of the M.I.s at $d=2$ and $d=4$ dimensions, as a simple but explicit example of functions exhibiting the properties described in the previous sections.

At $d=2$ the Cutkosky-Veltman rule [14, 15] gives for $S(d;p^2)$, as defined by the first of Eq.s(3.7),

$$\frac{1}{\pi} Im S(2;-W^2) = N_2 \int_{b_2}^{b_3} db \, \frac{1}{\sqrt{R_4(b;b_1,b_2,b_3,b_4)}} , \tag{7.1}$$

where the following notations were introduced:

$$\begin{aligned} &N_2 = 1/2 \\ &p^2 = -W^2, \qquad W \geq m_1+m_2+m_3 , \\ &(m_2-m_3)^2 = b_1 \leq (m_2+m_3)^2 = b_2 \leq (W-m_1)^2 = b_3 \leq (W+m_1)^2 = b_4 , \\ &R_4(b;b_1,b_2,b_3,b_4) = (b-b_1)(b-b_2)(b_3-b)(b_4-b) . \end{aligned} \tag{7.2}$$

We have the relation

$$R_4(b;b_1,b_2,b_3,b_4) = R_2(b,m_2^2,m_3^2)\; R_2(W^2,b,m_1^2) , \tag{7.3}$$

where

$$R_2(a,b,c) = a^2+b^2+c^2-2ab-2ac-2bc , \tag{7.4}$$

is the familiar invariant form appearing in the two-body phase space, showing that the system of the three particles, whose masses enter in the definition of $R_4(b;b_1,b_2,b_3,b_4)$, can be considered as the merging of a two-body system of total energy $\sqrt{b}$ and masses m_2, m_3 with a two-body system of total energy W and masses $\sqrt{b}, m_1$.

According to Eq.s(3.7), for i=1,2,3

$$\frac{1}{\pi} Im S_i(2;-W^2) = -\frac{d}{dm_i^2}\left(\frac{1}{\pi} Im S(2;-W^2)\right) ; \tag{7.5}$$

the integral representation Eq.(7.1), however, is of no use for obtaining $ImS_i(2;-W^2)$ through a direct differentiation (due to the appearance of end point singularities). It is more convenient to use Eq.(D.13) of the Appendix D, so that Eq.(7.1) becomes

$$\frac{1}{\pi}ImS(2;-W^2) = N_2\frac{2}{\sqrt{(b_4-b_2)(b_3-b_1)}}K(w^2)\ , \tag{7.6}$$

where $K(w^2)$ is the complete elliptic integral of the first kind, Eq.(D.6), and

$$\begin{aligned} w^2 &= \frac{(b_4-b_1)(b_3-b_1)}{(b_4-b_2)(b_3-b_1)} \\ &= \frac{(W+m_1+m_2-m_3)(W+m_1-m_2+m_3)(W-m_1+m_2-m_3)(W-m_1-m_2+m_3)}{(W+m_1+m_2+m_3)(W+m_1-m_2-m_3)(W-m_1+m_2-m_3)(W-m_1-m_2+m_3)}\ , \\ (b_4-b_2)(b_3-b_1) &= (W+m_1+m_2+m_3)(W+m_1-m_2-m_3) \\ &\quad\times(W-m_1+m_2-m_3)(W-m_1-m_2+m_3)\ . \end{aligned} \tag{7.7}$$

Let us observe, in passing, that, even if $ImS(2,-W^2)$ (and, more generally $S(d;p^2)$ as well) is obviously symmetric in the three masses m_1, m_2, m_3, the symmetry is not explicitly shown by the integral representation Eq.(7.1), while the manifest symmetry is restored in Eq.s(7.6,7.7).
One can now use Eq.(7.6) and Eq.(D.11) to carry out the derivative with respect to the masses m_i^2 in Eq.(7.5); the result reads

$$\begin{aligned} \frac{1}{\pi}ImS_1(2;-W^2) &= N_2\frac{1}{2m_1^2\sqrt{(b_3-b_1)(b_4-b_2)}}\frac{1}{(b_3-b_2)(b_4-b_1)} \\ &\quad\times\left[4m_1(m_1m_3^2+m_1m_2^2-m_1^3+2m_2m_3W+m_1W^2)\,K(w^2)\right. \\ &\quad\left.-P(-W^2,m_1,m_2,m_3)\,E(w^2)\right]\ , \end{aligned} \tag{7.8}$$

$$\begin{aligned} \frac{1}{\pi}ImS_2(2;-W^2) &= N_2\frac{1}{2m_2^2\sqrt{(b_3-b_1)(b_4-b_2)}}\frac{1}{(b_3-b_2)(b_4-b_1)} \\ &\quad\times\left[4m_2(m_2m_3^2+m_2m_1^2-m_2^3+2m_1m_3W+m_2W^2)\,K(w^2)\right. \\ &\quad\left.-P(-W^2,m_2,m_1,m_3)\,E(w^2)\right]\ , \end{aligned} \tag{7.9}$$

$$\begin{aligned} \frac{1}{\pi}ImS_3(2;-W^2) &= N_2\frac{1}{2m_3^2\sqrt{(b_3-b_1)(b_4-b_2)}}\frac{1}{(b_3-b_2)(b_4-b_1)} \\ &\quad\times\left[4m_3(m_3m_1^2+m_3m_2^2-m_3^3+2m_2m_1W+m_3W^2)\,K(w^2)\right. \\ &\quad\left.-P(-W^2,m_3,m_2,m_1)\,E(w^2)\right]\ , \end{aligned} \tag{7.10}$$

where w^2 is given by the first of Eq.s(7.7), $P(p^2,m_1,m_2,m_3)$ is the polynomial already introduced in Eq.(4.3), symmetric in the last two arguments, and $E(w^2)$ is the complete elliptic integral of the second kind, see Eq.(D.7).

Eq.s(7.6,7.8,7.9,7.10) express the four quantities $ImS(2;-W^2), ImS_i(2,-W^2), i=1,2,3$ in terms of just two functions, the elliptic integrals $K(w^2), E(w^2)$; therefore, the four imaginary parts cannot be all linearly independent. It is indeed easy to check that they satisfy the two equations

$$\begin{aligned} &-\frac{1}{12}(m_1^2-2m_2^2+m_3^2)\ ImS(2;-W^2)+\frac{1}{12}(-W^2+m_1^2-3m_2^2+3m_3^2)\ m_1^2\ ImS_1(2,-W^2) \\ &-\frac{1}{6}(-W^2+m_2^2)\ m_2^2\ ImS_2(2;-W^2)+\frac{1}{12}(-W^2+3m_1^2-3m_2^2+m_3^2)\ m_3^2\ ImS_3(2;-W^2)=0\ , \end{aligned}$$

$$\begin{aligned} &-\frac{1}{12}(m_1^2+m_2^2-2m_3^2)\ ImS(2;-W^2)+\frac{1}{12}(-W^2+m_1^2+3m_2^2-3m_3^2)\ m_1^2\ ImS_1(2;-W^2) \\ &+\frac{1}{12}(-W^2+3m_1^2+m_2^2-3m_3^2)\ m_2^2\ ImS_2(2;-W^2)-\frac{1}{6}(-W^2+m_3^2)\ m_3^2\ ImS_3(2;-W^2)=0\ , \end{aligned}$$

which are nothing but the imaginary parts of $Z_2(2;-W^2)$, $Z_3(2;-W^2)$, Eq.s(3.14,3.15).

As a further comment on the imaginary parts at $d=2$, let us observe that they take a finite value at threshold, *i.e.* in the $W \to (m_1+m_2+m_3)$ limit. In that limit, indeed, $b_3 \to b_2 = (m_2+m_3)^2$, and one finds

$$\int_{b_2}^{b_3} \frac{db}{\sqrt{R_4(b;b_1,b_2,b_3,b_4)}} \to \frac{1}{\sqrt{(b_2-b_1)(b_4-b_2)}} \int_{b_2}^{b_3} \frac{db}{\sqrt{(b-b_2)(b_3-b)}} = \frac{\pi}{\sqrt{(b_2-b_1)(b_4-b_2)}} \ ,$$

so that

$$\frac{1}{\pi} ImS(2;-W^2) \xrightarrow[W\to(m_1+m_2+m_3)]{} \frac{N_2}{4\sqrt{m_1m_2m_3(m_1+m_2+m_3)}} \ . \tag{7.11}$$

The extension to the value at threshold of the $ImS_i(2,-W^2)$ is similar, even if requiring one more term in the expansion due to the presence of the denominator $1/(b_3-b_2)$ in their definitions, Eq.s(7.8,7.9,7.10). The threshold values are

$$\begin{aligned}
&\frac{1}{\pi} ImS_1(2;-W^2) \xrightarrow[W\to(m_1+m_2+m_3)]{} \\
&\qquad\qquad \frac{N_2}{32}\left(-\frac{3}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}-\frac{1}{m_1+m_2+m_3}\right)\frac{1}{\sqrt{m_1m_2m_3(m_1+m_2+m_3)}} \ , \\
&\frac{1}{\pi} ImS_2(2;-W^2) \xrightarrow[W\to(m_1+m_2+m_3)]{} \\
&\qquad\qquad \frac{N_2}{32}\left(+\frac{1}{m_1}-\frac{3}{m_2}+\frac{1}{m_3}-\frac{1}{m_1+m_2+m_3}\right)\frac{1}{\sqrt{m_1m_2m_3(m_1+m_2+m_3)}} \ , \\
&\frac{1}{\pi} ImS_3(2;-W^2) \xrightarrow[W\to(m_1+m_2+m_3)]{} \\
&\qquad\qquad \frac{N_2}{32}\left(+\frac{1}{m_1}+\frac{1}{m_2}-\frac{3}{m_3}-\frac{1}{m_1+m_2+m_3}\right)\frac{1}{\sqrt{m_1m_2m_3(m_1+m_2+m_3)}} \ .
\end{aligned} \tag{7.12}$$

At $d=4$ the imaginary part of $S(d;p^2)$, by using the same notation as in Eq.(7.1), is given by

$$\frac{1}{\pi} ImS(4;-W^2) = N_4 \int_{b_2}^{b_3} db \ \frac{1}{b}\sqrt{R_4(b;b_1,b_2,b_3,b_4)} \ . \tag{7.13}$$

with

$$N_4 = \frac{1}{8W^2} \ . \tag{7.14}$$

At variance with the $d=2$ case, the $ImS_i(4,-W^2)$ can be obtained at once by differentiating with respect to the masses the previous integral representation for $ImS(4;-W^2)$. The result can be most conveniently expressed in terms of the four (independent) integrals $I(-1,W), I(0,W), I(1,W), I(2,W)$, defined (see Eq.(D.3) and the Appendix for more details and the relation to the standard complete elliptic integrals) through

$$I(n,W) = \int_{b_2}^{b_3} db \ b^n \frac{1}{\sqrt{R_4(b;b_1,b_2,b_3,b_4)}} \ . \tag{7.15}$$

An explicit calculation gives

$$\begin{aligned}
\frac{1}{\pi} ImS(4;-W^2) = N_4 \Big[& \ b_1b_2b_3b_4 \ I(-1,W) \\
& -\frac{3}{4}(b_2b_3b_4+b_1b_3b_4+b_1b_2b_4+b_1b_2b_3) \ I(0,W) \\
& +\frac{1}{2}(b_3b_4+b_2b_4+b_2b_3+b_1b_4+b_1b_3+b_1b_2) \ I(1,W) \\
& -\frac{1}{4}(b_1+b_2+b_3+b_4) \ I(2,W) \ \Big]
\end{aligned} \tag{7.16}$$

$$
\begin{aligned}
\frac{1}{\pi} ImS_1(4;-W^2) = N_4\Big[& b_1 b_2 \big(-(b_4-b_3)W + (b_4+b_3)m_1\big)\ I(-1,W) \\
& + \big((b_2+b_1)(b_4-b_3)W - (b_2 b_4 + b_2 b_3 + b_1 b_4 + b_1 b_3 + 2 b_1 b_2) m_1\big)\ I(0,W) \\
& + \big((b_4-b_3)W + (b_4+b_3+2b_2+2b_1)m_1\big)\ I(1,W) \\
& - 2m_1\ I(2,W)\ \Big]
\end{aligned}
\tag{7.17}
$$

$$
\begin{aligned}
\frac{1}{\pi} ImS_2(4;-W^2) = N_4\Big[& b_3 b_4 \big(-(b_2-b_1)m_3 + (b_2+b_1)m_2\big)\ I(-1,W) \\
& + \big(-(2b_3 b_4 + b_2 b_4 + b_2 b_3 + b_1 b_4 + b_1 b_3) m_2 + (b_2-b_1)(b_4+b_3)m_3\big)\ I(0,W) \\
& + \big((2b_4+2b_3+b_2+b_1)m_2 - (b_2-b_1)m_3\big)\ I(1,W) \\
& - 2m_2\ I(2,W)\ \Big]
\end{aligned}
\tag{7.18}
$$

$$
\begin{aligned}
\frac{1}{\pi} ImS_3(4;-W^2) = N_4\Big[& b_3 b_4 \big(-(b_2-b_1)m_2 + (b_2+b_1)m_3\big)\ I(-1,W) \\
& + \big(-(2b_3 b_4 + b_2 b_4 + b_2 b_3 + b_1 b_4 + b_1 b_3) m_3 + (b_2-b_1)(b_4+b_3)m_2\big)\ I(0,W) \\
& + \big((2b_4+2b_3+b_2+b_1)m_3 - (b_2-b_1)m_2\big)\ I(1,W) \\
& - 2m_3\ I(2,W)\ \Big]
\end{aligned}
\tag{7.19}
$$

Again at variance with the $d=2$ case, the four imaginary parts are now combinations of four independent elliptic integrals, and therefore all independent of each other.

Having recalled the main features of the imaginary parts of the M.I.s at $d=2$ and $d=4$ dimensions, we can look at the way the Tarasov's shifting relations work in their case.

Let us start from the "direct" shift expressing the imaginary parts at $d=2$ in terms of those at $d=4$. The $d\to 4$ limit of the shifting relations is trivial, even if the relevant formulas are as usual rather lengthy. Keeping only the imaginary parts of the master integrals one finds for the M.I. $S(2,p^2)$, with $-p^2 = W^2 \geq (m_1+m_2+m_3)^2$

$$
\begin{aligned}
\frac{1}{\pi} ImS(2,-W^2) = {} & \tilde{A}(W,m_1,m_2,m_3)\frac{1}{\pi} ImS(4,-W^2) \\
& + \tilde{B}(W,m_1,m_2,m_3)m_1 \frac{1}{\pi} ImS_1(4,-W^2) \\
& + \tilde{B}(W,m_2,m_3,m_1)m_2 \frac{1}{\pi} ImS_2(4,-W^2) \\
& + \tilde{B}(W,m_3,m_1,m_2)m_3 \frac{1}{\pi} ImS_3(4,-W^2)\ ,
\end{aligned}
\tag{7.20}
$$

where

$$
\begin{aligned}
\tilde{A}(W,m_1,m_2,m_3) &= A(W,m_1,m_2,m_3) + A(W,m_1,-m_2,m_3) \\
& \quad + A(W,m_1,m_2,-m_3) + A(W,m_1,-m_2,-m_3)\ , \\
\tilde{B}(W,m_1,m_2,m_3) &= B(W,m_1,m_2,m_3) + B(W,m_1,-m_2,m_3) \\
& \quad + B(W,m_1,m_2,-m_3) + B(W,m_1,-m_2,-m_3)\ , \\
A(W,m_1,m_2,m_3) &= \frac{1}{2m_1 m_2 m_3}\ \frac{m_1+m_2+m_3}{W^2-(m_1+m_2+m_3)^2}\ , \\
B(W,m_1,m_2,m_3) &= \frac{1}{2}(2m_1+m_2+m_3)A(W,m_1,m_2,m_3)\ .
\end{aligned}
\tag{7.21}
$$

Eq.(7.20) is relatively simple, and, when substituting in it the explicit values of $ImS(4,-W^2)$ and $ImS_i(4,-W^2)$, as given by Eq.s(7.16–7.19), Eq.(7.1) is recovered. The same happens for $ImS_i(2,-W^2)$, $i=1,2,3$ as well.

Conversely, one can look at the inverse formulas, giving the imaginary parts at $d+2 \to 4$ in terms of the imaginary parts at $d \to 2$. For $ImS(4, -W^2)$, taking only the imaginary part at $d = 2$ of Eq.(6.4), one obtains:

$$
\begin{aligned}
\frac{1}{\pi} ImS(4; -W^2) = C(-W^2, m_1, m_2, m_3) \frac{1}{\pi} ImS(2; -W^2) \\
+ C_1(-W^2, m_1, m_2, m_3) \frac{1}{\pi} ImS_1(2; -W^2) \\
+ C_2(-W^2, m_1, m_2, m_3) \frac{1}{\pi} ImZ_2^{(1)}(2; -W^2) \\
+ C_3(-W^2, m_1, m_2, m_3) \frac{1}{\pi} ImZ_3^{(1)}(2; -W^2) \ ,
\end{aligned}
\tag{7.22}
$$

where the $C(-W^2, m_1, m_2, m_3), C_i(-W^2, m_1, m_2, m_3)$ have been defined in the previous section, and their explicit expressions can be found in Eq.s(C.2-C.5), $ImS(2; -W^2), ImS_1(2; -W^2)$ are the imaginary parts of the corresponding Master Integrals at $d = 2$, while $ImZ_2^{(1)}(2; -W^2), ImZ_3^{(1)}(2; -W^2)$ are the imaginary parts of the first term of the expansion in $(d-2)$ of the corresponding functions, see Eq.s(4.11) (let us recall once more that according to Eq.s(3.14,3.15) $Z_2(2; p^2)$, $Z_3(2; p^2)$ vanish identically). An equation similar to Eq.(7.22) holds for $ImS_1(4; -W^2)$; we do not write it explicitly for the sake of brevity.
The functions $ImS(4; -W^2), ImS_1(4; -W^2)$ and $ImS(2; -W^2), ImS_1(2; -W^2)$ are known, see Eq.s(7.16,7.17) and Eq.s(7.6,7.8); by combining Eq.(7.22) and the similar (not written) equation for $ImS_1(4; -W^2)$, one can obtain the explicit values of $ImZ_2^{(1)}(2; -W^2), ImZ_3^{(1)}(2; -W^2)$. One finds

$$
\begin{aligned}
\frac{1}{\pi} ImZ_2^{(1)}(2; -W^2) = \frac{N_2}{16} \Big[(W^2 - m_3^2 + m_2^2 - m_1^2) I(0, W) + I(1, W) \\
- (m_3^2 - m_2^2)(W^2 - m_1^2) I(-1, W) \Big] \ ,
\end{aligned}
\tag{7.23}
$$

$$
\begin{aligned}
\frac{1}{\pi} ImZ_3^{(1)}(2; -W^2) = \frac{N_2}{16} \Big[(W^2 + m_3^2 - m_2^2 - m_1^2) I(0, W) + I(1, W) \\
+ (m_3^2 - m_2^2)(W^2 - m_1^2) I(-1, W) \Big] \ ,
\end{aligned}
\tag{7.24}
$$

From the previous equations and the same procedure giving Eq.s(7.11,7.12) we obtain in particular the values at threshold

$$
\begin{aligned}
\frac{1}{\pi} ImZ_2^{(1)}(2; -W^2) \xrightarrow[W \to (m_1+m_2+m_3)]{} \frac{N_2}{16} \sqrt{\frac{m_2(m_1+m_2+m_3)}{m_1 \, m_3}} \\
\frac{1}{\pi} ImZ_3^{(1)}(2; -W^2) \xrightarrow[W \to (m_1+m_2+m_3)]{} \frac{N_2}{16} \sqrt{\frac{m_3(m_1+m_2+m_3)}{m_1 \, m_2}} \ .
\end{aligned}
\tag{7.25}
$$

$ImZ_2^{(1)}(2; -W^2)$ can also be evaluated solving, by quadrature, the imaginary part of the differential equation Eq.(4.15), *i.e.* by evaluating

$$
ImZ_2^{(1)}(2; -W^2) = C + \int^{-W^2} dp^2 \ Im \left(\frac{d}{dp^2} Z_2^{(1)}(2; p^2) \right) \ ,
$$

where C is an integration constant and $dZ_2^{(1)}(2; p^2)/dp^2$ is obtained from Eq.(4.15) itself. The constant C can be fixed, *a posteriori*, by requiring that the imaginary parts of the "conventional" M.I. vanish at threshold in $d = 4$ dimensions, a condition which leads again to Eq.s(7.25).

After many algebraic simplifications, one obtains for $ImZ_2^{(1)}(2;-W^2)$

$$\begin{aligned} \frac{1}{\pi} ImZ_2^{(1)}(2;-W^2) = & \frac{N_2}{16}\sqrt{\frac{m_2(m_1+m_2+m_3)}{m_1m_3}} \\ & + \frac{1}{64}\int_{(m_1+m_2+m_3)^2}^{W^2} ds \Big[\ \tilde{F}(s,m_1,m_2,m_3) \ I(0,s) \\ & \qquad - \tilde{G}(s,m_1,m_2,m_3) \ I(1,s) \\ & \qquad + \tilde{H}(s,m_1,m_2,m_3) \ I(2,s) \Big] \ . \end{aligned} \tag{7.26}$$

where the three quantities $\tilde{F},\tilde{G},\tilde{H}$ are all expressed in terms of the corresponding functions F,G,H by the relation

$$\begin{aligned} \tilde{F}(s,m_1,m_2,m_3) = & F(s,m_1,m_2,m_3) + F(s,m_1,-m_2,m_3) \\ & + F(s,m_1,m_2,-m_3) + F(s,m_1,-m_2,-m_3) \ , \end{aligned}$$

and the explicit expressions of those functions are

$$\begin{aligned} F(s,m_1,m_2,m_3) &= \frac{(m_2+m_3)^2}{m_1m_3} \ \frac{2m_1^2+m_2^2+m_3^2+2m_1m_2+2m_1m_3}{s-(m_1+m_2+m_3)^2} \ , \\ G(s,m_1,m_2,m_3) &= 2\frac{m_1^2+m_2^2+m_3^2+m_1m_2+m_1m_3+m_2m_3}{m_1m_3\ [s-(m_1+m_2+m_3)^2]} \ , \\ H(s,m_1,m_2,m_3) &= \frac{1}{m_1m_3\ [s-(m_1+m_2+m_3)^2]} \ . \end{aligned}$$

To carry out the integration, we use the integral representations Eq.(D.4) for the elliptic integrals $I(n,s)$ and exchange the order of integration according to

$$\int_{(m_1+m_2+m_3)^2}^{W^2} ds \int_{(m_2+m_3)^2}^{(\sqrt{s}-m_1)^2} \frac{db}{\sqrt{R_4(b;\ b_1,b_2,b_3,b_4)}} = \int_{(m_2+m_3)^2}^{(W-m_1)^2} \frac{db}{\sqrt{R_2(b,m_2^2,m_3^2)}} \int_{(\sqrt{b}+m_1)^2}^{W^2} \frac{ds}{\sqrt{R_2(s,b,m_1^2)}} \ ,$$

where Eq.(7.3) was used. The s integration is then elementary, giving only logarithms of suitable arguments and new square roots quadratic in b; a subsequent integration by parts in b removes those logarithms with some of the accompanying square roots, and the result is Eq.(7.23), as expected.
The same applies also for $ImZ_3^{(1)}(2;-W^2)$, whose value is obtained by simply exchanging m_2 and m_3 in Eq.(7.23).

8 Conclusions

In this paper we introduced a new class of identities, dubbed Schouten identities, valid at fixed integer value of the dimensions d. We applied the identities valid at $d=2$ to the case of the massive two-loop sunrise graph with different masses, finding that in $d=2$ dimensions only two of the four Master Integrals (M.I.s) are actually independent, so that the other two can be expressed as suitable linear combinations of the latter.

In the general case of arbitrary dimension d and different masses, the four M.I.s are known to fulfil a system of four first-order coupled differential equations in the external momentum transfer. The system can equivalently be re-phrased as a fourth-order differential equation for one of the M.I.s only.

Using these relations we introduced a new set of four independent M.I.s, valid for any number of dimensions d, whose property is that two of the newly defined integrals vanish identically in $d=2$. The new system of differential equations for this set of M.I.s takes then a simpler block form when expanded in $(d-2)$.

Starting from this system, one can derive a second-order differential equation, *exact in* d, for the full scalar amplitude, which still contains the two integrals, whose value is zero at $d = 2$, as inhomogeneous terms. We verified that the zeroth-order of our equation corresponds to the equation derived in [9]. Our equations, once expanded in powers of $(d - 2)$, can be used, together with the linear equations for the remaining three M.I.s, for evaluating recursively, at least in principle, all four M.I.s, up to any order in $(d - 2)$.

We then worked out explicitly the Tarasov's shifting relations needed to recover the physically more relevant value of the four M.I.s expanded in $(d-4)$ at $d \approx 4$ starting from the expansion in $(d-2)$ at $d \approx 2$ worked out in our approach.

As an example of this procedure we discussed the relationship between the *imaginary parts* of the four M.I.s in $d = 2$ and $d = 4$. The latter can be computed using the Cutkosky-Veltman rule. We showed how in $d = 2$ the imaginary parts of the four M.I.s can be written in terms of *two* independent functions only, namely the complete elliptic integrals of the first and of the second kind. The same is not true in $d = 4$ dimensions, where four independent elliptic integrals are needed in order to represent the four imaginary parts. We then showed how the Tarasov's shift formulas relate the imaginary parts in $d = 2$ and $d = 4$ dimensions. Finally, we gave an explicit example of how the differential equations for the imaginary parts of the master integrals can be integrated by quadrature.

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A The polynomials of the first-order differential equations

In this appendix we give the explicit expressions for the polynomials appearing in the first-order differential equations in section 4. All polynomials are functions of p^2 and of the three masses m_1, m_2, m_3, while they do not depend on the dimensions d.

$$
\begin{aligned}
P_{10}^{(0)}(p^2, m_1, m_2, m_3) = & -m_1^2(m_1+m_2+m_3)(m_1-m_2+m_3)(m_1+m_2-m_3)(m_1-m_2-m_3) \\
& \times \left(m_1^2 - m_2^2 - m_3^2\right)^2 \\
& + p^2 \left(m_1^8 + 4m_2^2m_1^6 - 14m_2^4m_1^4 + 12m_2^6m_1^2 - 3m_2^8 + 4m_3^2m_1^6 + 4m_3^2m_2^4m_1^2 \right. \\
& \left. -8m_3^2m_2^6 - 14m_3^4m_1^4 + 4m_3^4m_2^2m_1^2 + 22m_3^4m_2^4 + 12m_3^6m_1^2 - 8m_3^6m_2^2 - 3m_3^8\right) \\
& + p^4 \left(10m_1^6 - 4m_2^2m_1^4 + 2m_2^4m_1^2 - 8m_2^6 - 4m_3^2m_1^4 + 16m_3^2m_2^4 + 2m_3^4m_1^2 \right. \\
& \left. +16m_3^4m_2^2 - 8m_3^6\right) \\
& + p^6 \left(14m_1^4 - 4m_2^2m_1^2 - 6m_2^4 - 4m_3^2m_1^2 + 24m_3^2m_2^2 - 6m_3^4\right) \\
& + 7p^8\, m_1^2 + p^{10}\,,
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
P_{10}^{(1)}(p^2,m_1,m_2,m_3) &= (m_1+m_2+m_3)(m_1-m_2+m_3)(m_1+m_2-m_3)(m_1-m_2-m_3) \\
&\times \left(m_1^2-m_2^2-m_3^2\right)\left(7m_1^4-6m_2^2m_1^2-m_2^4-6m_3^2m_1^2+2m_3^2m_2^2-m_3^4\right) \\
&+ p^2 \left(11m_1^8-48m_2^2m_1^6+78m_2^4m_1^4-56m_2^6m_1^2+15m_2^8-48m_3^2m_1^6\right. \\
&+68m_3^2m_2^2m_1^4-40m_3^2m_2^4m_1^2+20m_3^2m_2^6+78m_3^4m_1^4-40m_3^4m_2^2m_1^2 \\
&\left.-70m_3^4m_2^4-56m_3^6m_1^2+20m_3^6m_2^2+15m_3^8\right) \\
&+ p^4 \left(-2m_1^6-14m_2^2m_1^4+2m_2^4m_1^2+14m_2^6-14m_3^2m_1^4+60m_3^2m_2^2m_1^2\right. \\
&\left.-62m_3^2m_2^4+2m_3^4m_1^2-62m_3^4m_2^2+14m_3^6\right) \\
&- 2\,p^6\left(m_1^2-m_2^2-3m_3^2\right)\left(m_1^2-3m_2^2-m_3^2\right)+p^8\left(11m_1^2+m_2^2+m_3^2\right)+7\,p^{10}\,, \qquad \text{(A.2)}
\end{aligned}
$$

$$
\begin{aligned}
P_{10}^{(2)}(p^2,m_1,m_2,m_3) &= (m_1+m_2+m_3)(m_1-m_2+m_3)(m_1+m_2-m_3)(m_1-m_2-m_3) \\
&\times \left(5m_1^4-4m_2^2m_1^2-m_2^4-4m_3^2m_1^2+2m_3^2m_2^2-m_3^4\right) \\
&+ p^2 \left(8m_1^6-18m_2^2m_1^4+20m_2^4m_1^2-10m_2^6-18m_3^2m_1^4+24m_3^2m_2^2m_1^2\right. \\
&\left.+10m_3^2m_2^4+20m_3^4m_1^2+10m_3^4m_2^2-10m_3^6\right) \\
&+ p^4\left(10m_1^4+6m_2^2m_1^2-8m_2^4+6m_3^2m_1^2+48m_3^2m_2^2-8m_3^4\right) \\
&+ p^6\left(16m_1^2+10m_2^2+10m_3^2\right)+9\,p^8\,, \qquad \text{(A.3)}
\end{aligned}
$$

$$
\begin{aligned}
P_{11}^{(0)}(p^2,m_1,m_2,m_3) &= (m_1+m_2+m_3)^2(m_1-m_2+m_3)^2(m_1+m_2-m_3)^2(m_1-m_2-m_3)^2 \\
&\times \left(m_1^2-m_2^2-m_3^2\right)\, m_1^2 \\
&+ p^2\left(-6m_1^{10}+m_2^2m_1^8+32m_2^4m_1^6-42m_2^6m_1^4+14m_2^8m_1^2+m_2^{10}+m_3^2m_1^8\right. \\
&-64m_3^2m_2^2m_1^6+26m_3^2m_2^4m_1^4+40m_3^2m_2^6m_1^2-3m_3^2m_2^8+32m_3^4m_1^6 \\
&+26m_3^4m_2^2m_1^4-108m_3^4m_2^4m_1^2+2m_3^4m_2^6-42m_3^6m_1^4+40m_3^6m_2^2m_1^2 \\
&\left.+2m_3^6m_2^4+14m_3^8m_1^2-3m_3^8m_2^2+m_3^{10}\right) \\
&+ p^4\left(-33m_1^8+6m_2^2m_1^6-20m_2^4m_1^4+42m_2^6m_1^2+5m_2^8+6m_3^2m_1^6\right. \\
&-24m_3^2m_2^2m_1^4-26m_3^2m_2^4m_1^2-4m_3^2m_2^6-20m_3^4m_1^4-26m_3^4m_2^2m_1^2 \\
&\left.-2m_3^4m_2^4+42m_3^6m_1^2-4m_3^6m_2^2+5m_3^8\right) \\
&+ p^6\left(-52m_1^6-6m_2^2m_1^4+32m_2^4m_1^2+10m_2^6-6m_3^2m_1^4-64m_3^2m_2^2m_1^2\right. \\
&\left.+6m_3^2m_2^4+32m_3^4m_1^2+6m_3^4m_2^2+10m_3^6\right) \\
&+ p^8\left(-33m_1^4-m_2^2m_1^2+10m_2^4-m_3^2m_1^2+12m_3^2m_2^2+10m_3^4\right) \\
&+ p^{10}\left(-6m_1^2+5m_2^2+5m_3^2\right)+p^{12}\,, \qquad \text{(A.4)}
\end{aligned}
$$

$$
\begin{aligned}
P_{11}^{(1)}(p^2,m_1,m_2,m_3) &= (m_1+m_2+m_3)^2(m_1-m_2+m_3)^2(m_1+m_2-m_3)^2(m_1-m_2-m_3)^2 \\
&\times\left(5m_1^4-4m_2^2m_1^2-m_2^4-4m_3^2m_1^2+2m_3^2m_2^2-m_3^4\right) \\
&+p^2\left(2m_1^{10}-28m_2^2m_1^8+76m_2^4m_1^6-80m_2^6m_1^4+34m_2^8m_1^2\right. \\
&-4m_2^{10}-28m_3^2m_1^8+40m_3^2m_2^2m_1^6-48m_3^2m_2^4m_1^4+24m_3^2m_2^6m_1^2 \\
&+12m_3^2m_2^8+76m_3^4m_1^6-48m_3^4m_2^2m_1^4-116m_3^4m_2^4m_1^2-8m_3^4m_2^6 \\
&\left.-80m_3^6m_1^4+24m_3^6m_2^2m_1^2-8m_3^6m_2^4+34m_3^8m_1^2+12m_3^8m_2^2-4m_3^{10}\right) \\
&+p^4\left(-41m_1^8-42m_2^4m_1^4+88m_2^6m_1^2-5m_2^8+52m_3^2m_2^2m_1^4-152m_3^2m_2^4m_1^2\right. \\
&\left.+4m_3^2m_2^6-42m_3^4m_1^4-152m_3^4m_2^2m_1^2+2m_3^4m_2^4+88m_3^6m_1^2+4m_3^6m_2^2-5m_3^8\right) \\
&+p^6\left(-84m_1^6-8m_2^2m_1^4+60m_2^4m_1^2-8m_3^2m_1^4-184m_3^2m_2^2m_1^2+60m_3^4m_1^2\right) \\
&+p^8\left(-61m_1^4-8m_2^2m_1^2+5m_2^4-8m_3^2m_1^2+6m_3^2m_2^2+5m_3^4\right) \\
&+p^{10}\left(-14m_1^2+4m_2^2+4m_3^2\right)+p^{12}\,, \qquad \text{(A.5)}
\end{aligned}
$$

$$
\begin{aligned}
P_{12}^{(0)}(p^2,m_1,m_2,m_3) &= (m_1+m_2+m_3)(m_1-m_2+m_3)(m_1+m_2-m_3)(m_1-m_2-m_3) \\
&\times\left(m_1^2-m_2^2-m_3^2\right)\left(3m_1^2+m_2^2-m_3^2\right) \\
&+p^2\left(18m_1^6+2m_2^2m_1^4-10m_2^4m_1^2-10m_2^6-32m_3^2m_1^4+24m_3^2m_2^2m_1^2\right. \\
&\left.+40m_3^2m_2^4+34m_3^4m_1^2-10m_3^4m_2^2-20m_3^6\right) \\
&+p^4\left(36m_1^4-12m_2^2m_1^2-8m_2^4+6m_3^2m_1^2+66m_3^2m_2^2-34m_3^4\right) \\
&+p^6\left(30m_1^2+10m_2^2-4m_3^2\right)+9\,p^8\,, \qquad \text{(A.6)}
\end{aligned}
$$

$$
\begin{aligned}
P_{14}^{(1)}(p^2,m_1,m_2,m_3) &= (m_1+m_2+m_3)(m_1-m_2+m_3)(m_1+m_2-m_3)(m_1-m_2-m_3) \\
&\times\left(m_1^2-m_2^2-m_3^2\right)\left(m_1^2-m_2^2+m_3^2\right) \\
&+p^2\left(6m_1^6-22m_2^2m_1^4+26m_2^4m_1^2-10m_2^6+12m_3^2m_1^4+8m_3^2m_2^2m_1^2-20m_3^2m_2^4\right. \\
&\left.-18m_3^4m_1^2+30m_3^4m_2^2\right) \\
&+p^4\left(12m_1^4+8m_2^2m_1^2-20m_2^4-10m_3^2m_1^2+22m_3^2m_2^2+6m_3^4\right) \\
&+p^6\left(10m_1^2-6m_2^2+8m_3^2\right)+3\,p^8\,, \qquad \text{(A.7)}
\end{aligned}
$$

$$
\begin{aligned}
P_{14}^{(2)}(p^2,m_1,m_2,m_3) &= (m_1+m_2+m_3)(m_1-m_2+m_3)(m_1+m_2-m_3)(m_1-m_2-m_3) \\
&\times\left(m_1^2-m_2^2+m_3^2\right)\left(5m_1^4-4m_2^2m_1^2-m_2^4-4m_3^2m_1^2+2m_3^2m_2^2-m_3^4\right) \\
&+p^2\left(m_1^2-m_2^2+m_3^2\right)\left(17m_1^6-31m_2^2m_1^4+19m_2^4m_1^2-5m_2^6-29m_3^2m_1^4\right. \\
&\left.+46m_3^2m_2^2m_1^2+7m_3^2m_2^4+15m_3^4m_1^2+m_3^4m_2^2-3m_3^6\right) \\
&+p^4\left(22m_1^6+14m_2^2m_1^4-46m_2^4m_1^2+10m_2^6-42m_3^2m_1^4+72m_3^2m_2^2m_1^2\right. \\
&\left.-6m_3^2m_2^4+38m_3^4m_1^2-2m_3^4m_2^2-2m_3^6\right) \\
&+p^6\left(14m_1^4-16m_2^2m_1^2+10m_2^4+28m_3^2m_1^2+4m_3^2m_2^2+2m_3^4\right) \\
&+p^8\left(5m_1^2+5m_2^2+3m_3^2\right)+p^{10}\,, \qquad \text{(A.8)}
\end{aligned}
$$

$$
\begin{aligned}
P_{22}(p^2,m_1,m_2,m_3) &= 3m_1^4-2m_2^2m_1^2-m_2^4-2m_3^2m_1^2+2m_3^2m_2^2-m_3^4 \\
&+2\,p^2\left(m_1^2-m_2^2+m_3^2\right)+3\,p^4\,, \qquad \text{(A.9)}
\end{aligned}
$$

The polynomials defined above fulfil, among the others, the relation:

$$P_{14}^{(2)}(p^2,m_1,m_2,m_3) + P_{14}^{(2)}(p^2,m_1,m_3,m_2) - 2m_1^2 P_{10}^{(2)}(p^2,m_1,m_2,m_3) - 2p^2 P^2(p^2,m_1,m_2,m_3) = 0\,, \tag{A.10}$$

where note that the polynomial $P(p^2,m_1,m_2,m_3)$, defined in Eq.(4.3), appears squared.

B The polynomials of the second-order differential equation.

In this second appendix we give the explicit expressions of the polynomials that appear in the second-order differential equation derived in section 5. Also in this case, they are functions of p^2 and of the three masses m_1, m_2 and m_3, but they do not depend on the dimensions d.

$$\begin{aligned}
A_2^{(0)}(p^2,m_1,m_2,m_3) = &-(m_1-m_2-m_3)^3(m_1-m_2+m_3)^3(m_1+m_2-m_3)^3(m_1+m_2+m_3)^3 \\
&-8\,p^2(m_1-m_2-m_3)(m_1-m_2+m_3)(m_1+m_2-m_3)(m_1+m_2+m_3) \\
&\times\left(m_1^6-m_2^2m_1^4-m_2^4m_1^2+m_2^6-m_3^2m_1^4+10m_3^2m_2^2m_1^2\right. \\
&\left.-m_3^2m_2^4-m_3^4m_1^2-m_3^4m_2^2+m_3^6\right) \\
&-p^4\left(13m_1^8-36m_2^2m_1^6+46m_2^4m_1^4-36m_2^6m_1^2+13m_2^8-36m_3^2m_1^6\right. \\
&-124m_3^2m_2^2m_1^4-124m_3^2m_2^4m_1^2-36m_3^2m_2^6+46m_3^4m_1^4-124m_3^4m_2^2m_1^2 \\
&\left.+46m_3^4m_2^4-36m_3^6m_1^2-36m_3^6m_2^2+13m_3^8\right) \\
&+8\,p^6\left(m_1^2+m_2^2+m_3^2\right)\left(m_1^4+6m_2^2m_1^2+m_2^4+6m_3^2m_1^2+6m_3^2m_2^2+m_3^4\right) \\
&+p^8\left(37m_1^4+70m_2^2m_1^2+37m_2^4+70m_3^2m_1^2+70m_3^2m_2^2+37m_3^4\right) \\
&+32\,p^{10}\left(m_1^2+m_2^2+m_3^2\right)+9\,p^{12}
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
A_2^{(1)}(p^2,m_1,m_2,m_3) = &-\frac{1}{2}(m_1-m_2-m_3)^3(m_1-m_2+m_3)^3(m_1+m_2-m_3)^3(m_1+m_2+m_3)^3 \\
&+p^2(m_1-m_2-m_3)(m_1-m_2+m_3)(m_1+m_2-m_3)(m_1+m_2+m_3) \\
&\times\left(5m_1^6-5m_2^2m_1^4-5m_2^4m_1^2+5m_2^6-5m_3^2m_1^4+2m_3^2m_2^2m_1^2\right. \\
&\left.-5m_3^2m_2^4-5m_3^4m_1^2-5m_3^4m_2^2+5m_3^6\right) \\
&+\frac{1}{2}p^4\left(41m_1^8-84m_2^2m_1^6+86m_2^4m_1^4-84m_2^6m_1^2+41m_2^8-84m_3^2m_1^6\right. \\
&+52m_3^2m_2^2m_1^4+52m_3^2m_2^4m_1^2-84m_3^2m_2^6+86m_3^4m_1^4+52m_3^4m_2^2m_1^2 \\
&\left.+86m_3^4m_2^4-84m_3^6m_1^2-84m_3^6m_2^2+41m_3^8\right) \\
&+2p^6\left(11m_1^6-19m_2^2m_1^4-19m_2^4m_1^2+11m_2^6-19m_3^2m_1^4+54m_3^2m_2^2m_1^2\right. \\
&\left.-19m_3^2m_2^4-19m_3^4m_1^2-19m_3^4m_2^2+11m_3^6\right) \\
&+\frac{1}{2}p^8\left(m_1^4-50m_2^2m_1^2+m_2^4-50m_3^2m_1^2-50m_3^2m_2^2+m_3^4\right) \\
&-11p^{10}\left(m_1^2+m_2^2+m_3^2\right)-\frac{9}{2}p^{12}
\end{aligned} \tag{B.2}$$

$$
\begin{aligned}
A_3^{(0)}(p^2,m_1,m_2,m_3) =& (m_1-m_2-m_3)(m_1-m_2+m_3)(m_1+m_2-m_3)(m_1+m_2+m_3) \\
&\times \left(m_1^6-m_2^2m_1^4-m_2^4m_1^2+m_2^6-m_3^2m_1^4\right. \\
&\left.+6m_3^2m_2^2m_1^2-m_3^2m_2^4-m_3^4m_1^2-m_3^4m_2^2+m_3^6\right) \\
&+p^2\left(5m_1^8-8m_2^2m_1^6+6m_2^4m_1^4-8m_2^6m_1^2+5m_2^8-8m_3^2m_1^6\right. \\
&-8m_3^2m_2^2m_1^4-8m_3^2m_2^4m_1^2-8m_3^2m_2^6+6m_3^4m_1^4-8m_3^4m_2^2m_1^2 \\
&\left.+6m_3^4m_2^4-8m_3^6m_1^2-8m_3^6m_2^2+5m_3^8\right) \\
&+2p^4\left(3m_1^6-7m_2^2m_1^4-7m_2^4m_1^2+3m_2^6-7m_3^2m_1^4\right. \\
&\left.-7m_3^2m_2^4-7m_3^4m_1^2-7m_3^4m_2^2+3m_3^6\right) \\
&-2p^6\left(m_1^4+8m_2^2m_1^2+m_2^4+8m_3^2m_1^2+8m_3^2m_2^2+m_3^4\right) \\
&-7p^8\left(m_1^2+m_2^2+m_3^2\right)-3p^{10}
\end{aligned} \tag{B.3}
$$

$$
\begin{aligned}
A_3^{(1)}(p^2,m_1,m_2,m_3) =& -\frac{1}{2}(m_1-m_2-m_3)(m_1-m_2+m_3)(m_1+m_2-m_3)(m_1+m_2+m_3) \\
&\times \left(m_1^2-m_2^2-m_3^2\right)\left(m_1^2-m_2^2+m_3^2\right)\left(m_1^2+m_2^2-m_3^2\right) \\
&-\frac{1}{2}p^2\left(17m_1^8-32m_2^2m_1^6+18m_2^4m_1^4-8m_2^6m_1^2+5m_2^8-32m_3^2m_1^6\right. \\
&+20m_3^2m_2^2m_1^4+8m_3^2m_2^4m_1^2+4m_3^2m_2^6+18m_3^4m_1^4+8m_3^4m_2^2m_1^2 \\
&\left.-18m_3^4m_2^4-8m_3^6m_1^2+4m_3^6m_2^2+5m_3^8\right) \\
&-p^4\left(21m_1^6-31m_2^2m_1^4+7m_2^4m_1^2+3m_2^6-31m_3^2m_1^4+30m_3^2m_2^2m_1^2\right. \\
&\left.-3m_3^2m_2^4+7m_3^4m_1^2-3m_3^4m_2^2+3m_3^6\right) \\
&-p^6\left(17m_1^4-20m_2^2m_1^2-m_2^4-20m_3^2m_1^2+22m_3^2m_2^2-m_3^4\right) \\
&-\frac{1}{2}p^8\left(5m_1^2-7m_2^2-7m_3^2\right)+\frac{3}{2}p^{10}
\end{aligned} \tag{B.4}
$$

$$
\begin{aligned}
A_4(p^2,m_1,m_2,m_3) =& -\left(m_1^2-m_2^2\right)\Big[\\
&24(m_1-m_2-m_3)(m_1-m_2+m_3)(m_1+m_2-m_3)(m_1+m_2+m_3) \\
&\times\left(m_1^2+m_2^2-m_3^2\right) \\
&+8p^2\left(9m_1^4-10m_2^2m_1^2+9m_2^4-14m_3^2m_1^2-14m_3^2m_2^2+5m_3^4\right) \\
&+24p^4\left(3m_1^2+3m_2^2-7m_3^2\right)+24p^6\Big]
\end{aligned} \tag{B.5}
$$

$$
\begin{aligned}
A_5^{(1)}(p^2,m_1,m_2,m_3) =& (m_1-m_2-m_3)(m_1-m_2+m_3)(m_1+m_2-m_3)(m_1+m_2+m_3) \\
&\times\left(m_1^4-2m_2^2m_1^2+m_2^4+m_3^2m_1^2+m_3^2m_2^2-2m_3^4\right) \\
&+p^2\left(3m_1^6-3m_2^2m_1^4-3m_2^4m_1^2+3m_2^6-8m_3^2m_1^4-8m_3^2m_2^4\right. \\
&\left.+11m_3^4m_1^2+11m_3^4m_2^2-6m_3^6\right) \\
&+p^4\left(3m_1^4-14m_2^2m_1^2+3m_2^4+7m_3^2m_1^2+7m_3^2m_2^2-6m_3^4\right) \\
&+p^6\left(m_1^2+m_2^2-2m_3^2\right)
\end{aligned} \tag{B.6}
$$

$$
\begin{aligned}
A_5^{(2)}(p^2,m_1,m_2,m_3) = &-\frac{1}{2}\,(m_1-m_2-m_3)(m_1-m_2+m_3)(m_1+m_2-m_3)(m_1+m_2+m_3)\\
&\times\left(m_1^2-m_2^2-m_3^2\right)\left(m_1^2-m_2^2+m_3^2\right)\\
&-p^2\left(3m_1^6-3m_2^2m_1^4-3m_2^4m_1^2+3m_2^6-2m_3^2m_1^4+4m_3^2m_2^2m_1^2\right.\\
&\left.-2m_3^2m_2^4+7m_3^4m_1^2+7m_3^4m_2^2-8m_3^6\right)\\
&-p^4\left(6m_1^4-12m_2^2m_1^2+6m_2^4+11m_3^2m_1^2+11m_3^2m_2^2-13m_3^4\right)\\
&-p^6\left(5m_1^2+5m_2^2-4m_3^2\right)-\frac{3}{2}\,p^8
\end{aligned}
\tag{B.7}
$$

Note that, in order to derive Eq.(5.2), we made use of the following relation:

$$
A_5^{(1)}(p^2,m_1,m_2,m_3)+A_5^{(1)}(p^2,m_1,m_3,m_2)+A_5^{(1)}(p^2,m_2,m_3,m_1)=0\,. \tag{B.8}
$$

C Tarasov's shift

In this Appendix we enclose the explicit formula for the order zero of the Tarasov's shift, Eq.(6.4) discussed in section 6, which relates the zeroth-order of the full scalar amplitude, evaluated in $d=4$ dimensions, to a linear combination of the four new M.I.s evaluate in $d=2$ dimensions, namely $S(2;p^2)$, $S_1(2;p^2)$, $Z_2^{(1)}(2;p^2)$ and $Z_3^{(1)}(2;p^2)$.

$$
\begin{aligned}
S^{(0)}(4;p^2) = &-\frac{1}{128}\left[13p^2+24(m_1^2+m_2^2+m_3^2)\right]\\
&+\frac{1}{8}\Big[(m_1^2-m_2^2-m_3^2)\ln(m_3)\ln(m_2)-(m_1^2-m_2^2+m_3^2)\ln(m_3)\ln(m_1)\\
&-(m_1^2+m_2^2-m_3^2)\ln(m_2)\ln(m_1)-\ln(m_3)^2m_3^2-\ln(m_2)^2m_2^2-\ln(m_1)^2m_1^2\Big]\\
&+\frac{1}{96\,p^2}\bigg\{\Big[2p^4+6(4m_1^2+m_2^2+m_3^2)p^2\\
&+(2m_1^4-6m_2^2m_1^2-m_2^4-6m_3^2m_1^2+12m_3^2m_2^2-m_3^4)\Big]\ln(m_1)\\
&+\Big[2p^4+6(m_1^2+4m_2^2+m_3^2)p^2\\
&-(m_1^4+6m_2^2m_1^2-2m_2^4-12m_3^2m_1^2+6m_3^2m_2^2+m_3^4)\Big]\ln(m_2)\\
&+\Big[2p^4+6(m_1^2+m_2^2+4m_3^2)p^2\\
&-(m_1^4-12m_2^2m_1^2+m_2^4+6m_3^2m_1^2+6m_3^2m_2^2-2m_3^4)\Big]\ln(m_3)\Big]\bigg\}\\
&-\frac{1}{96\,p^2\,P(p^2,m_1,m_2,m_3)}\bigg\{\Big[2p^8-2(2m_1^2-5m_2^2-5m_3^2)p^6\\
&+(26m_1^4-56m_2^2m_1^2+13m_2^4-56m_3^2m_1^2+32m_3^2m_2^2+13m_3^4)p^4\\
&+2(16m_1^6-25m_2^2m_1^4-17m_2^4m_1^2+2m_2^6-25m_3^2m_1^4+8m_3^2m_2^4\\
&-17m_3^4m_1^2+8m_3^4m_2^2+2m_3^6)p^2\\
&+(16m_2^2m_1^6-13m_2^4m_1^4-2m_2^6m_1^2-m_2^8+16m_3^2m_1^6-100m_3^2m_2^2m_1^4+22m_3^2m_2^4m_1^2\\
&+14m_3^2m_2^6-13m_3^4m_1^4+22m_3^4m_2^2m_1^2-26m_3^4m_2^4-2m_3^6m_1^2+14m_3^6m_2^2-m_3^8)\Big]\ln(m_1)
\end{aligned}
$$

$$
\begin{aligned}
&- \Big[p^8 - 2(m_1^2 - 7m_2^2 + 2m_3^2)p^6 \\
&+ (13m_1^4 - 22m_2^2m_1^2 + 26m_2^4 - 34m_3^2m_1^2 + 16m_3^2m_2^2 - 13m_3^4)p^4 \\
&+ 2(8m_1^6 - 50m_2^2m_1^4 + 11m_2^4m_1^2 + 7m_2^6 + 25m_3^2m_1^4 - 8m_3^2m_2^4 \\
&- 28m_3^4m_1^2 + 16m_3^4m_2^2 - 5m_3^6)p^2 \\
&+ (-16m_2^2m_1^6 + 13m_2^4m_1^4 + 2m_2^6m_1^2 + m_2^8 + 32m_3^2m_1^6 - 50m_3^2m_2^2m_1^4 - 34m_3^2m_2^4m_1^2 \\
&+ 4m_3^2m_2^6 - 26m_3^4m_1^4 + 56m_3^4m_2^2m_1^2 - 13m_3^4m_2^4 - 4m_3^6m_1^2 + 10m_3^6m_2^2 - 2m_3^8)\Big] \ln(m_2) \\
&- \Big[p^8 - 2(m_1^2 + 2m_2^2 - 7m_3^2)p^6 \\
&+ (13m_1^4 - 34m_2^2m_1^2 - 13m_2^4 - 22m_3^2m_1^2 + 16m_3^2m_2^2 + 26m_3^4)p^4 \\
&+ 2(8m_1^6 + 25m_2^2m_1^4 - 28m_2^4m_1^2 - 5m_2^6 - 50m_3^2m_1^4 + 16m_3^2m_2^4 \\
&+ 11m_3^4m_1^2 - 8m_3^4m_2^2 + 7m_3^6)p^2 \\
&- (-32m_2^2m_1^6 + 26m_2^4m_1^4 + 4m_2^6m_1^2 + 2m_2^8 + 16m_3^2m_1^6 + 50m_3^2m_2^2m_1^4 - 56m_3^2m_2^4m_1^2 \\
&- 10m_3^2m_2^6 - 13m_3^4m_1^4 + 34m_3^4m_2^2m_1^2 + 13m_3^4m_2^4 - 2m_3^6m_1^2 - 4m_3^6m_2^2 - m_3^8)\Big] \ln(m_3)\Big\} \\
&- \frac{1}{4\,p^2\,P(p^2,m_1,m_2,m_3)}\Big\{(3m_1^2 - 2m_2^2 - 2m_3^2)p^8 \\
&+ 2(2m_2^2m_1^2 - m_2^4 + 2m_3^2m_1^2 - 8m_3^2m_2^2 - m_3^4)p^6 \\
&- 2(5m_1^6 - 8m_2^2m_1^4 + 8m_2^4m_1^2 - m_2^6 - 8m_3^2m_1^4 \\
&+ 2m_3^2m_2^2m_1^2 + 5m_3^2m_2^4 + 8m_3^4m_1^2 + 5m_3^4m_2^2 - m_3^6)p^4 \\
&- 2(4m_1^8 - 6m_2^2m_1^6 + 3m_2^4m_1^4 - m_2^8 - 6m_3^2m_1^6 + 8m_3^2m_2^4m_1^2 \\
&+ 6m_3^2m_2^6 + 3m_3^4m_1^4 + 8m_3^4m_2^2m_1^2 - 10m_3^4m_2^4 + 6m_3^6m_2^2 - m_3^8)p^2 \\
&- (m_1 - m_2 - m_3)(m_1 - m_2 + m_3)(m_1 + m_2 - m_3)(m_1 + m_2 + m_3) \\
&\times (m_1^2 - m_2^2 - m_3^2)(m_1^2 + m_2^2 + m_3^2)m_1^2\Big\} S^{(0)}(2;p^2) \\
&+ \frac{D(p^2,m_1,m_2,m_3)}{4\,p^2\,P(p^2,m_1,m_2,m_3)}\left[p^2 - (m_1^2 + m_2^2 + m_3^2)\right] S_1^{(0)}(2;p^2)m_1^2 \\
&- \frac{4}{p^2}\Big\{(p^2 + m_3^2)(m_1^2 - m_2^2)Z_2^{(1)}(2;p^2) + (p^2 + m_2^2)(m_1^2 - m_3^2)Z_3^{(1)}(2;p^2)\Big\}, \qquad \text{(C.1)}
\end{aligned}
$$

where $P(p^2, m_1, m_2, m_3)$ and $D(p^2, m_1, m_2, m_3)$ are the usual polynomials defined in Eq.s(4.3,4.9).

In particular, from this equation we can read off the explicit values of the functions $C(p^2, m_1, m_2, m_3)$, $C_1(p^2, m_1, m_2, m_3)$, $C_2(p^2, m_1, m_2, m_3)$ and $C_3(p^2, m_1, m_2, m_3)$ introduced in Eq.s(6.4,7.22):

$$
\begin{aligned}
C(p^2,m_1,m_2,m_3) = &- \frac{1}{4\,p^2\,P(p^2,m_1,m_2,m_3)}\Big[(3m_1^2 - 2m_2^2 - 2m_3^2)p^8 \\
&+ 2(2m_2^2m_1^2 - m_2^4 + 2m_3^2m_1^2 - 8m_3^2m_2^2 - m_3^4)p^6 \\
&- 2(5m_1^6 - 8m_2^2m_1^4 + 8m_2^4m_1^2 - m_2^6 - 8m_3^2m_1^4 \\
&+ 2m_3^2m_2^2m_1^2 + 5m_3^2m_2^4 + 8m_3^4m_1^2 + 5m_3^4m_2^2 - m_3^6)p^4 \\
&- 2(4m_1^8 - 6m_2^2m_1^6 + 3m_2^4m_1^4 - m_2^8 - 6m_3^2m_1^6 + 8m_3^2m_2^4m_1^2 \\
&+ 6m_3^2m_2^6 + 3m_3^4m_1^4 + 8m_3^4m_2^2m_1^2 - 10m_3^4m_2^4 + 6m_3^6m_2^2 - m_3^8)p^2 \\
&- (m_1 - m_2 - m_3)(m_1 - m_2 + m_3)(m_1 + m_2 - m_3)(m_1 + m_2 + m_3) \\
&\times (m_1^2 - m_2^2 - m_3^2)(m_1^2 + m_2^2 + m_3^2)m_1^2\Big], \qquad \text{(C.2)}
\end{aligned}
$$

$$C_1(p^2, m_1, m_2, m_3) = \frac{m_1^2\, D(p^2, m_1, m_2, m_3)}{4\, p^2\, P(p^2, m_1, m_2, m_3)} \left[p^2 - (m_1^2 + m_2^2 + m_3^2)\right] , \tag{C.3}$$

$$C_2(p^2, m_1, m_2, m_3) = -\frac{4}{p^2}(p^2 + m_3^2)(m_1^2 - m_2^2) , \tag{C.4}$$

$$C_3(p^2, m_1, m_2, m_3) = -\frac{4}{p^2}(p^2 + m_2^2)(m_1^2 - m_3^2) . \tag{C.5}$$

D Imaginary parts

We work out here in some details the formulas used in Section 7. To start with, let us recall the definitions Eq.s(7.2)

$$\begin{aligned} &(m_2 - m_3)^2 = b_1 \leq (m_2 + m_3)^2 = b_2 \leq (W - m_1)^2 = b_3 \leq (W + m_1)^2 = b_4 \ , \\ &R_4(b; b_1, b_2, b_3, b_4) = (b - b_1)(b - b_2)(b_3 - b)(b_4 - b) \ , \end{aligned} \tag{D.1}$$

with

$$W \geq (m_1 + m_2 + m_3) \ . \tag{D.2}$$

Let us define

$$I(n, W) = \int_{b_2}^{b_3} db\ b^n \frac{1}{\sqrt{R_4(b;\ b_1, b_2, b_3, b_4)}} \ . \tag{D.3}$$

One has obviously

$$\int_{b_2}^{b_3} db \frac{d}{db}\left[b^n \sqrt{R_4(b; b_1, b_2, b_3, b_4)}\right] = 0 \ ;$$

by working out the derivative, one gets an identity involving up to five integrals of the type $I(n, W)$ with different values of n; one finds that they can all be expressed as combination of four of them, which can be chosen to be

$$\begin{aligned} I(-1, W) &= \int_{b_2}^{b_3} \frac{db}{b\, \sqrt{R_4(b;\ b_1, b_2, b_3, b_4)}} , \\ I(0, W) &= \int_{b_2}^{b_3} \frac{db}{\sqrt{R_4(b;\ b_1, b_2, b_3, b_4)}} , \\ I(1, W) &= \int_{b_2}^{b_3} \frac{db\ b}{\sqrt{R_4(b;\ b_1, b_2, b_3, b_4)}} , \\ I(2, W) &= \int_{b_2}^{b_3} \frac{db\ b^2}{\sqrt{R_4(b;\ b_1, b_2, b_3, b_4)}} . \end{aligned} \tag{D.4}$$

In the same way, starting for instance from

$$\int_{b_2}^{b_3} db \frac{d}{db}\left[\frac{1}{b - b_1} \sqrt{R_4(b; b_1, b_2, b_3, b_4)}\right] = 0 \ ,$$

one finds

$$\begin{aligned} \int_{b_2}^{b_3} \frac{db}{(b - b_1)\, \sqrt{R_4(b;\ b_1, b_2, b_3, b_4)}} &= \frac{1}{(b_2 - b_1)(b_3 - b_1)(b_4 - b_1)} \\ &\times \left[b_1(b_1 - b_2 - b_3 - b_4)I(0, W) + (b_1 - b_2 - b_3 - b_4)I(1, W) - 2I(2, W)\right] \ . \end{aligned} \tag{D.5}$$

The above four integrals $I(n,W)$, with $n=-1,0,1,2$, defined in Eq.s(D.4), are easily expressed in terms of the usual complete elliptic integrals $K(w^2), E(w^2), \Pi(a;w^2)$ of first, second and third kind, namely:

$$K(w^2) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-w^2\,x^2)}}\,, \qquad 0 < w^2 < 1\,, \tag{D.6}$$

$$E(w^2) = \int_0^1 dx \sqrt{\frac{1-w^2x^2}{1-x^2}}\,, \qquad 0 < w^2 < 1\,, \tag{D.7}$$

$$\Pi(a;w^2) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-w^2\,x^2)}\;(1-a\,x^2)}\,, \qquad 0 < w^2, a < 1\,. \tag{D.8}$$

Indeed, the standard change of variable

$$b = \frac{b_1(b_3-b_2)x^2 - b_2(b_3-b_1)}{(b_3-b_2)x^2-(b_3-b_1)}\,, \quad x^2 = \frac{(b_3-b_1)(b-b_2)}{(b_3-b_2)(b-b_1)}\,, \tag{D.9}$$

gives

$$\begin{aligned}
I(-1,W) &= \frac{2}{\sqrt{(b_3-b_1)(b_4-b_2)}}\frac{1}{b_1b_2}\left[b_2\ K(w^2) - (b_2-b_1)\ \Pi(a_1,w^2)\right]\,, \\
I(0,W) &= \frac{2}{\sqrt{(b_3-b_1)(b_4-b_2)}}\,K(w^2)\,, \\
I(1,W) &= \frac{2}{\sqrt{(b_3-b_1)(b_4-b_2)}}\left[b_1\ K(w^2) + (b_2-b_1)\ \Pi(a_1,w^2)\right]\,, \\
I(2,W) &= \frac{2}{\sqrt{(b_3-b_1)(b_4-b_2)}}\Big[(b_1^2+b_1(b_2+b_3)-b_2b_3)\ K(w^2) - (b_3-b_1)(b_4-b_2)\ E(w^2) \\
&\qquad +(b_2-b_1)(b_1+b_2+b_3+b_4)\ \Pi(a_2,w^2)\Big]\,,
\end{aligned} \tag{D.10}$$

where

$$\begin{aligned}
w^2 &= \frac{(b_4-b_1)(b_3-b_2)}{(b_4-b_2)(b_3-b_1)}\,, \\
a_1 &= \frac{b_1(b_3-b_2)}{b_2(b_3-b_1)}\,, \\
a_2 &= \frac{(b_3-b_2)}{(b_3-b_1)}\,.
\end{aligned}$$

With the integral representation of Eq.s(D.6,D.7) it is easy to obtain the formula

$$\frac{d}{dw^2}K(w^2) = \frac{1}{2w^2}\left[\frac{E(w^2)}{1-w^2} - K(w^2)\right]\,, \tag{D.11}$$

which is useful for the evaluation of the imaginary parts of the Master Integrals $S_i(d,p^2)$ in $d=2$ dimensions, Eq.(7.5).

One can easily express also $K(w^2), E(w^2)$ etc. in terms of the $I(n,W)$ by inverting Eq.s(D.10) or by using the change of variables Eq.(D.9). Indeed, the second of Eq.s(D.10) can also be written as

$$K(w^2) = \frac{1}{2}\sqrt{(b_3-b_1)(b_4-b_2)}\ I(0,W)\,, \tag{D.12}$$

or, recalling the definition of $I(0,W)$, Eq.(D.4),

$$\int_{b_2}^{b_3} \frac{db}{\sqrt{R_4(b;\,b_1,b_2,b_3,b_4)}} = \frac{2}{\sqrt{(b_3-b_1)(b_4-b_2)}}\ K(w^2)\,, \tag{D.13}$$

while the change of variable $x \to b$ gives

$$E(w^2) = \frac{1}{2}\sqrt{(b_3-b_1)(b_4-b_2)}\ \frac{b_2-b_1}{b_4-b_2}\int_{b_2}^{b_3} \frac{db}{\sqrt{R_4(b;\,b_1,b_2,b_3,b_4)}}\ \frac{b_4-b}{b-b_1}\ ,$$

which on account of Eq.(D.5) can also be written as

$$E(w^2) = -\frac{1}{2\sqrt{(b_3-b_1)(b_4-b_2)}}\Big[(b_2b_3+b_1b_4)\ I(0,E) - (b_1+b_2+b_3+b_4)\ I(1,E) + 2\ I(2,E)\Big]\ . \quad \text{(D.14)}$$

Let us discuss shortly also the limit of equal masses $m_1 = m_2 = m_3 = m$, which gives $b_1 = 0, b_2 = 4m^2, b_3 = (W-m)^2, b_4 = (W+m)^2$. In that limit, thanks to $b_1 = 0$ we can read Eq.(D.5) as an identity expressing $I(-1,W)$ in terms of $I(1,W), I(2,W)$. Further, one more identity appears [16] from

$$\int_{4m^2}^{(W-m)^2} db\,\frac{d}{db}\ \ln\left(\frac{b(W^2+3m^2-b)+\sqrt{R_2(b,m^2,m^2)}\sqrt{R_2(W^2,b,m^2)}}{b(W^2+3m^2-b)-\sqrt{R_2(b,m^2,m^2)}\sqrt{R_2(W^2,b,m^2)}}\right) = 0\ ,$$

where $R_2(a,b,c)$ is defined in Eq.(7.4), giving (in the equal mass limit),

$$I(1,W) = \frac{1}{3}(W^2+3m^2)\ I(0,W)\ , \quad \text{(D.15)}$$

showing once more that in the equal mass limit the imaginary parts can be expressed in terms of two independent functions only.

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